Definition

An ordinary differential equation

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \ldots + a_1 x' + a_0 x = f,$$

with coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ independent of t is called an **ODE with constant coefficients**.

Example

$$y''' + y'' = x$$

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Remark

The coefficient f is in general a function, also in the case of an ODE with constant coefficients.

Plan

Consider a homogenous linear differential equation of order n, i.e., an equation of the form

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \ldots + a_1 x' + a_0 x = 0, \quad (1)$$

where $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$.

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where $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$. It follows that if $x = e^{\lambda t}$ is a solution of this equation, then λ is a root of the polynomial

$$P(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \ldots + a_1 \lambda + a_0.$$

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We call $P(\lambda)$ the characteristic polynomial of the equation (1).

Exercise

Prove the above statement.

The polynomial $P(\lambda)$ has n complex roots, from which a fundamental system of (1) can be constructed.

The polynomial $P(\lambda)$ has n complex roots, from which a fundamental system of (1) can be constructed. Recall that, since $P(\lambda)$ has real coefficients, if $\lambda_1 = \alpha + \beta i \in \mathbb{C}$ is a root of $P(\lambda)$, then so is its conjugate $\lambda_2 = \overline{\lambda_1} = \alpha - \beta i \in \mathbb{C}$.

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2. If $\lambda \in \mathbb{R}$ is a root of multiplicity k, then

$$x_0(t) = e^{\lambda t},\ x_1(t) = te^{\lambda t}, \dots,\ x_{k-1}(t) = t^{k-1}e^{\lambda t}$$

are solutions of (1).

3. If $\lambda = \alpha + \beta i \in \mathbb{C}$ is a simple root, then

$$x_0(t) = e^{\alpha t} \sin \beta t, \;\; x_1(t) = e^{\alpha t} \cos \beta t$$

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are solutions of (1).

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$$\begin{array}{rclrcl} x_0(t) & = & e^{\alpha t} \sin \beta t, & x_1(t) & = & e^{\alpha t} \cos \beta t, \\ x_2(t) & = & t e^{\alpha t} \sin \beta t, & x_3(t) & = & t e^{\alpha t} \cos \beta t, , \\ & \vdots & & & \vdots & \\ x_{2k-2}(t) & = & t^{k-1} e^{\alpha t} \sin \beta t, & x_{2k-1}(t) & = & t^{k-1} e^{\alpha t} \cos \beta t, \end{array}$$

are solutions of (1).

Theorem

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of $P(\lambda)$, then the corresponding functions $x_1, x_2, \ldots, x_n$ defined as above form a fundamental system of (1).

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In particular, the general solution of (1) is given by

$$x(t) = C_1 x_1(t) + C_2 x_2(t) + \ldots + C_n x_n(t),$$

where $C_1, C_2, \ldots, C_n \in \mathbb{R}$.

Plan

Consider an inhomogeneous linear differential equation of order n, i.e. an equation of the form

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \ldots + a_1 x' + a_0 x = f(t), \quad (2)$$

where $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$ and f is a function.

It is known that the **general solution** of this equation is of the form

$$x = x_h + x_p,$$

where x_h is the **general solution** of the corresponding homogenous equation

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \ldots + a_1 x' + a_0 x = 0 \quad (3)$$

and x_p is a solution of (2), called a **particular solution**.

We find x_h as in the previous section.

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The table below should be read as follows: If $f(t)$ has a form as in the left column, then the particular solution $x_p(t)$ has a form given in the right column.

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$f(t)$	$x_p(t)$
polynomial p_n of order n $P(0) \neq 0$	$q_n(t)$ – polynomial fo order n
polynomial p_m of order m 0 is a k-fold root of $P(\lambda)$	$t^k \cdot q_n(t)$
$ke^{\gamma t}$, $P(\gamma) \neq 0$	$Ce^{\gamma t}$
$ke^{\gamma t}$ γ is a k-fold root of $P(\lambda)$	$Ct^k e^{\gamma t}$
$a \cos \omega t + b \sin \omega t$ $P(\pm \omega i) \neq 0$	$A \cos \omega t + B \sin \omega t$
$a \cos \omega t + b \sin \omega t$ $\pm \omega i$ a k-fold root of P	$At^k \cos \omega t + Bt^k \sin \omega t$

P – characteristic polynomial

If x_p given by the table is a solution of the homogenous equation (3), then it obviously cannot be a solution of the inhomogeneous equation (2).

If x_p given by the table is a solution of the homogenous equation (3), then it obviously cannot be a solution of the inhomogeneous equation (2). In order to find a particular solution of the inhomogeneous equation in such case, we modify x_p by multiplying it by t^k, where k is the smallest number such that $t^k x_p$ is not a solution of (3).

If $f(t)$ is a sum of elements from the left column, the particular solution x_p is the sum of the corresponding elements from the right column.