Higher order ordinary differential equations

27 października 2020

The equation of the form

$$a_n(t)x^{(n)}+a_{n-1}(t)x^{(n-1)}+a_{n-2}(t)x^{(n-2)}+\ldots+a_1(t)x'+a_0(t)x=f(t),$$

where $a_n, a_{n-1}, \ldots, a_1, a_0, f$ are given functions of t (defined on an interval (a, b) with values in $\mathbb{R}$) and $x', \ldots, x^{(n-1)}, x^{(n)}$ are the successive derivatives of the unknown function x of the variable t, is called a **linear differential equation of order** n.

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ODE of 2nd order:

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If there are conditions of the form:

$$\begin{cases} x(t_0) & = & x_0 \\ x'(t_0) & = & x_1 \\ & \vdots & \\ x^{(n-2)}(t_0) & = & x_{n-2} \\ x^{(n-1)}(t_0) & = & x_{n-1}, \end{cases}$$

where $t_0 \in (a, b)$ and $x_0, x_1, \ldots x_{n-1} \in \mathbb{R}$, then the equation with these conditions is called an **initial value problem**.

Example

$$x'' - 4x' + 4x = e^{2t}$$
$$x(0) = 1$$
$$x'(0) = 0$$

Consider a homogenous equation:

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + a_{n-2}(t)x^{(n-2)} + \ldots + a_1(t)x' + a_0(t)x = 0. \tag{1}$$

Definition

The sequence $(x_1(t), x_2(t), \ldots, x_n(t))$ of solutions of (1) is called a **fundamental system** of this equation if for every $t \in (a, b)$ it follows that

$$\det \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \\ x_1'(t) & x_2'(t) & \ldots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t) & x_2^{(n-1)}(t) & \ldots & x_n^{(n-1)}(t) \end{bmatrix} \neq 0$$

The determinant given in the above definition is called the **Wronskian** of $(x_1(t), x_2(t), \ldots, x_n(t))$.

Theorem

If $(x_1(t), x_2(t), \ldots, x_n(t))$ *is a fundamental system of* (1)*, then the general solution of* (1) *is given by*

$$x(t) = C_1x_1(t) + C_2x_2(t) + \ldots + C_nx_n(t),$$

where $C_1, C_2, \ldots, C_n \in \mathbb{R}$*.*

In other words, all the solutions of (1) are of the above form.

Example

$$(1 - t^2)x'' - 2tx' + 2x = 0$$

$$FS : x_1(t) = t$$

$$x_2(t) = \frac{t}{2} \cdot \ln \frac{1+t}{1-t} - 1$$

Example

$$x''' - x'' - 8x' + 12x = 0$$

$$FS: x_1(t) = e^{-3t}$$

$$x_2(t) = e^{2t}$$

$$x_3(t) = te^{2t}$$

Example

$$x'' - 4x' + 4x = 0$$

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