Calculus – repetition

6 października 2020

Derivatives of basic functions:

$$
\begin{aligned}
(c)' &= 0, \\
(x)' &= 1, \\
(x^a)' &= a \cdot x^{a-1}, \\
(e^x)' &= e^x, \\
(a^x)' &= a^x \ln a, \qquad && a \in \mathbb{R}_+ \setminus \{1\}, \\
(\ln |x|)' &= \frac{1}{x}, \qquad && x \in \mathbb{R}_+, \\
(\log_a x)' &= \frac{1}{x \ln a}, \qquad && a \in \mathbb{R}_+ \setminus \{1\},
\end{aligned}
$$

$$
\begin{aligned}
(\sin x)' &= \cos x, \\
(\cos x)' &= -\sin x, \\
(\tan x)' &= \frac{1}{\cos^2 x} = 1 + \tan^2 x, \qquad x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}, \\
(\cot x)' &= -\frac{1}{\sin^2 x} = -(1 + \cot^2 x), \qquad x \neq \pi + k\pi,\ k \in \mathbb{Z}, \\
(\arcsin x)' &= \frac{1}{\sqrt{1 - x^2}}, \qquad x \in (-1, 1), \\
(\arccos x)' &= \frac{-1}{\sqrt{1 - x^2}}, \qquad x \in (-1, 1), \\
(\arctan x)' &= \frac{1}{1 + x^2}, \\
(\operatorname{arccot} x)' &= -\frac{1}{1 + x^2}.
\end{aligned}
$$

Theorem

Let f and g be differentiable functions. Then,

1. $(f+g)'(x) = f'(x) + g'(x)$ (i.e., differentiation is **additive**),
2. $(f-g)'(x) = f'(x) - g'(x)$,
3. $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$,
4. for x such that $g(x) \neq 0$,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}.$$

Example

Let c be a real constant, and f a differentiable function. Then,

$$[(cf)(x)]' = c' \cdot f(x) + c \cdot f'(x) = 0 \cdot f(x) + c \cdot f'(x) = c \cdot f'(x).$$

It can be understood as a next differentiation rule, so-called **homogeneity**.

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Let $f(x) \neq 0$. Then,

$$\left(\frac{1}{f(x)}\right)' = \frac{1' \cdot f(x) - 1 \cdot f'(x)}{[f(x)]^2} = \frac{-f'(x)}{[f(x)]^2}.$$

It can be understood as a next differentiation rule.

Example

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\sin' x \cdot \cos x - \sin x \cdot \cos' x}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x + \sin x \cdot \sin x}{\cos^2 x} = \frac{1}{\cos^2 x},$$

$$x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$$

Theorem

(Compound function) If a compund function $f \circ g$ is defined in a neighbourhood of a point x_0, the function g is differentiable in x_0, and f is differentiable in $g_0 = g(x_0)$, then the derivative of the compund function $f \circ g$ in x_0 is given by

$$(f \circ g)'(x_0) = f'\left(g(x_0)\right) \cdot g'(x_0) = f'(g_0) \cdot g'(x_0).$$

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It can be written as

$$\frac{d(f \circ g)}{dx}(x_0) = \frac{df}{dg}(g_0) \cdot \frac{dg}{dx}(x_0).$$

Example

$$\left(e^{1/x}\right)' = e^{1/x} \cdot \frac{-1}{x^2} = \frac{-e^{1/x}}{x^2}, \qquad x \neq 0$$

Theorem

(Derivative of the inverse function) If a differentiable functiona f has an inverse function f^{-1}, then the derivative of the inverse function is equal to the inverse derivative of the original function:

$$\left(f^{-1}(x)\right)' = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

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Remark

It can be written as

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Example

1. Derivative of logarithm:

$$(\ln x)' = \frac{1}{(e^y)'}\bigg|_{y=\ln x} = \frac{1}{e^y}\bigg|_{y=\ln x} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

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2. Derivative of arcus cosine:

$$(\arccos x)' = \frac{1}{(\cos y)'}\bigg|_{y=\arccos x} = \frac{1}{-\sin y}\bigg|_{y=\arccos x}$$

$$\overset{(*)}{=} \frac{-1}{\sqrt{1-\cos^2 y}}\bigg|_{y=\arccos x} = \frac{-1}{\sqrt{1-x^2}}$$

$(*)$ – in the domain (the set, in this case an interval, where the function is defined) of arcus cosine, i.e., $(0, \pi)$, If f' is differentiable, then the second derivative of f is defined by $f''(x) = [f'(x)]'$. Further derivatives are defined similarly.

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Example

$$\begin{aligned} f(x) &= -4x^3 + 2x, \\ f'(x) &= -12x^2 + 2, \\ f''(x) &= -24x, \\ f'''(x) &= -24, \\ f^{(4)}(x) &= 0. \end{aligned}$$

Example

$$f(x) = \cos \ln x, \qquad x \in \mathbb{R}_+,$$

$$f'(x) = -\sin \ln x \cdot \frac{1}{x} = \frac{-\sin \ln x}{x},$$

$$f''(x) = \frac{\left(-\cos \ln x \cdot \frac{1}{x}\right) \cdot x - (-\sin \ln x) \cdot 1}{x^2}$$

$$= \frac{\sin \ln x - \cos \ln x}{x^2} = -\frac{f(x)}{x^2} - \frac{f'(x)}{x}.$$

Basic rules of integration:

1. $\int x^a\,dx = \frac{x^{a+1}}{a+1} + C$, dla $a \neq -1$, $x \in \mathbb{R}_+$ (since $\left(\frac{x^{a+1}}{a+1} + C\right)' = \frac{(a+1)\cdot x^a}{a+1} = x^a$)

   If a is a positive integer, then $x \in \mathbb{R}$; if it is a negative integer, then $x \neq 0$.

Example

Several special cases:

- $\int dx = x + C$
- $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$, $x \in \mathbb{R}_+$
- $\int \frac{dx}{x^2} = -\frac{1}{x} + C$, $x \neq 0$

2. $\int \frac{dx}{x} = \ln|x| + C,\ x \neq 0$ (bo $(\ln x)' = \frac{1}{x}$, $(\ln(-x))' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$)
3. $\int e^x dx = e^x + C$
4. $\int a^x dx = \frac{a^x}{\ln a} + C,\ a \in \mathbb{R} \setminus \{1\}$
5. $\int \sin x\, dx = -\cos x + C$
6. $\int \cos x\, dx = \sin x + C$
7. $\int \frac{dx}{\cos^2 x} = \tan x + C,\ \cos x \neq 0$
8. $\int \frac{dx}{\sin^2 x} = -\cot x + C,\ \sin x \neq 0$
9. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C_1 = -\arccos x + C_2,\ -1 < x < 1$
10. $\int \frac{dx}{x^2+1} = \operatorname{arc\,tg} x + C_1 = -\operatorname{arc\,ctg} x + C_2$

Example

Integrals of polynomials:

$$\int \sum_{k=0}^{n} \left(a_k x^k\right) dx = \sum_{k=0}^{n} \int a_k x^k dx$$

$$= \sum_{k=0}^{n} a_k \int x^k dx = \sum_{k=0}^{n} \frac{a_k x^{k+1}}{k+1} + C$$

1. $\int (2x^2 - 3x + 1) dx = \frac{2}{3}x^3 - \frac{3}{2}x^2 + x + C,$
2. $\int (7x^6 - 6x^5 + 5x^4 - 4x^3 + 3x^2 - 2x + 1) dx = x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x + C,$
3. $\int (3x^3 + x^2 - x - 1) dx = \frac{3x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} - x + C.$

Theorem

(Partial integraltion) If the functions u and v have continuous derivatives, then

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

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Proof. It follows from differential calculus that

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x).$$

Integration on both sides and subtraction of $\int u'v$ yields the desired formula.

Example

$$\int x^2 \ln x \, dx = \left[\begin{array}{c|c} u = \ln x & u' = 1/x \\ v' = x^2 & v = x^3/3 \end{array} \right] = \frac{x^3}{3} \ln x - \int \frac{x^3}{3x} \, dx$$

$$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$$

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$$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$$

Remark

As long as there is an undefined integral in the expression, there is no need to write the constant C, since it is included in the integral. After the last integral is computed, one cannot forget the constant.

Example

$$\int x^2 \cos x \, dx = \left[\begin{array}{c|c} u = x^2 & u' = 2x \\ v' = \cos x & v = \sin x \end{array} \right]$$

$$= x^2 \sin x - 2 \int x \sin x \, dx = \left[\begin{array}{c|c} u = x & u' = 1 \\ v' = \sin x & v = -\cos x \end{array} \right]$$

$$= x^2 \sin x - 2 \left[-x \cos x - \int (-\cos x) dx \right]$$

$$= x^2 \sin x + 2x \cos x - 2 \int \cos x \, dx$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

Example

$$\int e^x \cos x \, dx = \left[\begin{array}{c|c} u = e^x & u' = e^x \\ v' = \cos x & v = \sin x \end{array} \right]$$

$$= e^x \sin x - \int e^x \sin x \, dx = \left[\begin{array}{c|c} u = e^x & u' = e^x \\ v' = \sin x & v = -\cos x \end{array} \right]$$

$$= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx \right]$$

$$= e^x (\sin x + \cos x) - \int e^x \cos x \, dx$$

Example

The integral of $e^x \cos x$ occurs on both sides, so its value can be computed from the equation

$$\int e^x \cos x \, dx = e^x(\sin x + \cos x) - \int e^x \cos x \, dx.$$

Remember that an integral is defined up to a constant, so it can have different values on the left and on the right side of the equation.

Example

We write it as

$$\int e^x \cos x \, dx = e^x(\sin x + \cos x) - \int e^x \cos x \, dx + C.$$

Thus, we get

$$\int e^x \cos x \, dx = \frac{e^x(\sin x + \cos x)}{2} + C.$$

Note that C is a different constant than in the previous equation.

Remark

By partial integration, the function with a simple primitive, e.g., sine, cosine, exponential function, or sometimes power function (see the first example) will be chosen as v'. If there are monomials, we usually try to make their degree lower.

Theorem

(Substitution) If

1. function $f : I \to \mathbb{R}$ is continuous in interval I,
2. function $u : J \to I$ has a continuous derivative in interval J,

then

$$\int f\left(u(x)\right) u'(x)\, dx = \int f(t) dt = F(u(x)) + C,$$

where F is any primitive function of f.

Example

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \left[\begin{array}{c} u = \cos x \\ u' = -\sin x \end{array} \right] =$$

$$- \int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C, \qquad x \neq \frac{\pi}{2} + k\pi, \; k \in \mathbb{Z}$$

Example

$$\int \frac{2x}{\sqrt{x^2-1}}\,dx = \begin{bmatrix} u = x^2 - 1 \\ du = 2x\,dx \end{bmatrix} = \int \frac{du}{\sqrt{u}}$$
$$= 2\sqrt{u} + C = 2\sqrt{x^2-1} + C,$$

$x^2 - 1 > 0.$

Remark

Note the difference in notation when u is introduced. In the second example we use the differential of u, $du = u' dx$.

Example

$$\int \frac{dx}{\sqrt{2x-3}} = \left[\begin{array}{c} t = \sqrt{2x-3} \\ t^2 = 2x-3 \\ 2t\,dt = 2dx \end{array}\right] = \int \frac{t\,dt}{t}$$

$$= \int dt = t + C = \sqrt{2x-3} + C,$$

$x > \frac{3}{2}$,

Example

or with another substitution:

$$\int \frac{dx}{\sqrt{2x-3}} = \left[\begin{array}{c} t = 2x - 3 \\ dt = 2dx \end{array} \right] = \frac{1}{2} \int t^{-\frac{1}{2}} dt$$

$$= \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{2x-3} + C,$$

$x > \frac{3}{2}$.

Example

$$\int x^2\sqrt{2x^3-3}\,dx = \begin{bmatrix} t=\sqrt{2x^3-3} \\ t^2=2x^3-3 \\ 2t\,dt=6x^2\,dx \end{bmatrix} = \int t\cdot\frac{t}{3}\,dt = \frac{1}{3}\int t^2\,dt$$

$$= \frac{1}{3}\cdot\frac{t^3}{3}+C = \frac{1}{9}\left(\sqrt{2x^3-3}\right)^3+C, \qquad x\geq\sqrt[3]{\frac{3}{2}}$$

Example

or with another substitution:

$$\int x^2\sqrt{2x^3-3}\,dx = \left[\begin{array}{l} t = 2x^3-3 \\ dt = 6x^2\,dx \end{array}\right] = \frac{1}{6}\int t^{\frac{1}{2}}\,dt = \frac{1}{6}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{1}{9}\left(\sqrt{t}\right)^3 + C = \frac{1}{9}\left(\sqrt{2x^3-3}\right)^3 + C, \qquad x \geq \sqrt[3]{\frac{3}{2}}$$