Macierze i układy równań[1]

1. *Obliczyć*

(a) $\begin{bmatrix}1 & 3\\ -2 & 1\end{bmatrix} + 2\begin{bmatrix}2 & 1\\ 0 & 4\end{bmatrix}$, (b) $3\begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 1\end{bmatrix} - \begin{bmatrix}3 & 1 & 0\\ 1 & 1 & -1\end{bmatrix}$,

(c) $\begin{bmatrix}3 & -4 & 5\\ 2 & -3 & 1\\ 3 & -5 & -1\end{bmatrix}\begin{bmatrix}3 & 29\\ 2 & 18\\ 0 & -3\end{bmatrix}$, (d) $\begin{bmatrix}1 & -1 & 1 & -1\\ -1 & 1 & -1 & 1\end{bmatrix}\begin{bmatrix}0\\ -1\\ 2\\ -3\end{bmatrix}$,

(e) $2\begin{bmatrix}1 & 3\\ -2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}2 & -1\\ 1 & -1\\ -2 & 3\end{bmatrix}^T + 2\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$, (f) $2\begin{bmatrix}1 & 3\\ -2 & 1\\ 1 & -1\end{bmatrix}^T\begin{bmatrix}2 & -1\\ 1 & -1\\ -2 & 3\end{bmatrix} - 3\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$,

(g) $\begin{bmatrix}1 & 3\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 1\\ 0 & 1\\ 1 & 0\end{bmatrix}^T\begin{bmatrix}1\\ -1\\ 0\\ 1\end{bmatrix}\begin{bmatrix}3\end{bmatrix}$, (h) $\begin{bmatrix}1 & -1\\ -1 & 1\\ 0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\ -1 & 1\end{bmatrix}^T - 2\left(\begin{bmatrix}1\\ 2\end{bmatrix}\begin{bmatrix}1\\ -1\\ 0\\ 1\end{bmatrix}^T\right)^T$.

2. *Obliczyć następujące wyznaczniki*

(a) $\begin{vmatrix}1 & 2\\ 3 & -1\end{vmatrix}$, (b) $\begin{vmatrix}2 & 5\\ -1 & -1\end{vmatrix}$, (c) $\begin{vmatrix}a & 1\\ -3 & -1-a\end{vmatrix}$, (d) $\begin{vmatrix}1 & 2 & 3\\ -3 & -1 & 0\\ 1 & 2 & -1\end{vmatrix}$,

(e) $\begin{vmatrix}1 & 2a & 3+a\\ -3 & -1+a & 0\\ 1-a & 2 & -1\end{vmatrix}$, (f) $\begin{vmatrix}3 & 1 & 0\\ -2 & 5 & -1\\ 4 & 0 & -1\end{vmatrix}$, (g) $\begin{vmatrix}1 & -2 & 4\\ 1 & 1 & 1\\ 1 & 2 & 4\end{vmatrix}$, (h) $\begin{vmatrix}1 & -1 & 2 & 0\\ 0 & 1 & 0 & -3\\ 3 & 2 & -2 & 4\\ 2 & 3 & 1 & 1\end{vmatrix}$,

(i) $\begin{vmatrix}0 & 1 & -2 & 1\\ 2 & -1 & 3 & 0\\ 0 & 2 & -1 & 3\\ 1 & -3 & -1 & 2\end{vmatrix}$, (j) $\begin{vmatrix}1 & 0 & 0 & 1\\ 1 & 0 & 1 & 0\\ 1 & 1 & 0 & 0\\ 1 & 3 & 9 & 27\end{vmatrix}$, (k) $\begin{vmatrix}2 & -1 & 2 & 1\\ 0 & 1 & -2 & -3\\ 3 & 1 & -2 & -2\\ -2 & 3 & 1 & 3\end{vmatrix}$, (l) $\begin{vmatrix}x & 2 & 1 & -1\\ -2 & 1 & 3 & -3\\ -1 & x & -2 & 4\\ 2 & 3 & -2 & 1\end{vmatrix}$,

(m) $\begin{vmatrix}1 & -1 & 2 & 1 & -1\\ 1 & -2 & 2 & 1 & -3\\ -2 & 0 & 1 & 2 & 2\\ 2 & -1 & 1 & -3 & -2\\ 2 & 1 & -1 & 3 & 1\end{vmatrix}$, (n) $\begin{vmatrix}1 & 2 & 0 & 3 & -2\\ -2 & 2 & 1 & -4 & 3\\ 2 & -1 & 0 & 3 & 1\\ -1 & 1 & -2 & 0 & 3\\ 2 & 1 & -1 & -3 & 2\end{vmatrix}$, (o) $\begin{vmatrix}-1 & 1 & x & 0 & 1\\ 2 & 2 & 2 & 4 & 3\\ 1 & 0 & 1 & 3 & 2\\ -2 & 1 & -1 & -3 & x\\ x & 1 & 1 & 3 & 1\end{vmatrix}$, (p) $\begin{vmatrix}1 & 0 & q & 3 & 2\\ 2 & 2 & q & 4 & 3\\ 2 & 1 & 1 & 3 & 1\\ -1 & -1 & q & 0 & 1\\ -2 & 1 & -1 & -3 & -2\end{vmatrix}$.

3. *Obliczyć macierze odwrotne do następujących (o ile to możliwe):*

(a) $\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}$, (b) $\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}$, (c) $\begin{bmatrix}2 & 1\\ -4 & -2\end{bmatrix}$, (d) $\begin{bmatrix}-2 & 1\\ 3 & -2\end{bmatrix}$, (e) $\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 2\\ 1 & 2 & -1\end{bmatrix}$,

(f) $\begin{bmatrix}1 & 0 & -1\\ -2 & -1 & 0\\ 1 & 0 & -1\end{bmatrix}$, (g) $\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & -1\\ 1 & 0 & -1\end{bmatrix}$, (h) $\begin{bmatrix}1 & -1 & 2\\ -2 & 1 & -1\\ 1 & 2 & -1\end{bmatrix}$, (i) $\begin{bmatrix}1 & 0 & 0 & 1\\ 1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}$, (j) $\begin{bmatrix}1 & 2 & 1 & 1\\ -1 & -1 & 1 & 0\\ 0 & 1 & 2 & 0\\ 1 & 3 & 9 & 1\end{bmatrix}$.

4. *Wyznaczyć macierz X z równania*

(a) $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}X\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix} = \begin{bmatrix}5 & 6\\ -1 & 0\end{bmatrix}$, (b) $X\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 3\end{bmatrix} = \begin{bmatrix}1 & 1 & 4\\ 1 & 4 & 1\\ 1 & 1 & 4\end{bmatrix}$, (c) $\begin{bmatrix}1 & 1 & -1\\ 2 & 1 & 0\\ 1 & -1 & 1\end{bmatrix}X = \begin{bmatrix}1 & -1 & 3\\ 4 & 3 & 2\\ 1 & -2 & 5\end{bmatrix}$,

(d) $\begin{bmatrix}1 & 0 & 1\\ 0 & -1 & 1\\ 1 & -1 & 0\end{bmatrix}X = \begin{bmatrix}1 & -1\\ 1 & -2\\ 0 & 1\end{bmatrix}$, (e) $X\begin{bmatrix}1 & 2 & 1\\ -2 & -1 & 1\\ 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & -1\end{bmatrix}^T$, (f) $X\begin{bmatrix}1 & 0 & 1\\ 2 & -1 & 1\\ 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 1 & -2\\ 2 & -1 & 0\end{bmatrix}$,

(g) $X\begin{bmatrix}1 & -1 & 1\\ 1 & 2 & 0\\ 0 & -1 & 1\end{bmatrix} = \begin{bmatrix}2 & 1 & -1\\ 1 & -1 & 0\end{bmatrix}$, (h) $\begin{bmatrix}2 & 0 & 1\\ 1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}X = \begin{bmatrix}2 & 0\\ 1 & -1\\ 0 & 1\end{bmatrix}$, (i) $2X\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 2\\ 1 & 0 & -1\end{bmatrix} = 3\begin{bmatrix}-1 & 1\\ 0 & -1\\ 2 & 1\end{bmatrix}^T$.

5. *Rozwiązać następujące układy równań przy użyciu wzorów Cramera:*

(a) $\left\{\begin{aligned}2x-5y &= 2\\ x-3y &= 5,\end{aligned}\right.$ (b) $\left\{\begin{aligned}y+2x &= 1\\ x &= 2y+x,\end{aligned}\right.$ (c) $\left\{\begin{aligned}3x+2y+z &= 5\\ 2x+3y+z &= 1\\ 2x+y+3z &= 11,\end{aligned}\right.$

(d) $\left\{\begin{aligned}x-y+3z &= 8\\ x+z &= -1\\ x+y+z &= 2,\end{aligned}\right.$ (e) $\left\{\begin{aligned}y+2z-1 &= x\\ z+2x &= 1+2y\\ y-z &= -1,\end{aligned}\right.$ (f) $\left\{\begin{aligned}x-2z+1 &= y\\ z+2x-y &= 2z\\ x+y-z &= -1,\end{aligned}\right.$

(g) $\left\{\begin{aligned}t+u-1 &= w\\ w-3t &= 1+2u\\ u+2w &= -1+t,\end{aligned}\right.$ (h) $\left\{\begin{aligned}x &= 4y+1\\ 2x-2z &= 1+y\\ 3y+2x &= 2z,\end{aligned}\right.$ (i) $\left\{\begin{aligned}2x+y-z+2t &= 2\\ y+z-t &= 0\\ x-3y-2z+t &= 1\\ -x+y-2z-t &= -1.\end{aligned}\right.$

[1]wersja z 23 września 2019

6. *Korzystając z twierdzenia Kroneckera-Capelliego, sprawdzić ile rozwiązań mają poniższe układy. Jeżeli nie są sprzeczne, rozwiązać je metodą eliminacji Gaussa:*

$$(a)\ \begin{cases} x-3y+6z+4t = 5 \\ -x+2y-4z-3t = -4 \\ x+y-2z = 1 \\ 2x+3y-6z-t = 1, \end{cases} \quad (b)\ \begin{cases} x-2y+7z-4t = 1 \\ 2x+y-z-3t = 2 \\ -x-3y+8z-t = -1 \\ x-3y+10z-5t = 1, \end{cases} \quad (c)\ \begin{cases} -2x-3y-z = -1 \\ -x+y-2z+3t = -3 \\ 2y-3z+6t = -2 \\ 2x-y+3z-4t = 5, \end{cases}$$

$$(d)\ \begin{cases} x-2y+2z+3t = 5 \\ -x+2y+z = -5 \\ y-2z-2t = -2 \\ 2x+3z+5t = 2, \end{cases} \quad (e)\ \begin{cases} x+3z-2t = 3 \\ x-y-z-t = -1 \\ 2x-y+2z-3t = 2 \\ 3x-y+5z-5t = -5, \end{cases} \quad (f)\ \begin{cases} 2x-3y+2z-t = 15 \\ -2x+3y+t = -9 \\ 3x-y-2z-5t = 4 \\ 2y+3z-2t = 7, \end{cases}$$

$$(g)\ \begin{cases} x+5y+8z+6t = -2 \\ -x-y-z-t = 1 \\ -2x+z+t = 1 \\ x+2y+3z+2t = -1, \end{cases} \quad (h)\ \begin{cases} x+y+z-3t+u = 3 \\ 3x-2y+t+5u = -2 \\ -y+2z+2t+3u = 3 \\ 2x+y-4t+u = 1, \end{cases} \quad (i)\ \begin{cases} x-3y+2z+3t = 0 \\ 2x-y-2t = 1 \\ y-z-5t = 0 \\ -3x+z+2t = -2, \end{cases}$$

$$(j)\ \begin{cases} 3x-5y+2z+4t = 2 \\ 7x-4y+z+3t = 5 \\ 5x+7y-4z-6t = 3, \end{cases} \quad (k)\ \begin{cases} x-2y+7z-4t = 1 \\ 2x+y-z-3t = 2 \\ -x-3y+8z-t = -1 \\ x-3y+10z-5t = 1, \end{cases} \quad (l)\ \begin{cases} x+y-3z+t = 3 \\ x-y+z+t = 1 \\ -2x+y-2t = -3 \\ x+2y-5z+t = 4, \end{cases}$$

$$(m)\ \begin{cases} 2x+y = t+z \\ y-x = t+z-3 \\ t-y = 2-z \\ 2t+2z = 2y+x+3, \end{cases} \quad (n)\ \begin{cases} 2x-y+3z = 9 \\ 3x-5y+z = 2 \\ 4x-7y+z = 5, \end{cases} \quad (o)\ \begin{cases} x-2y-5z = -5 \\ 2x-3y-8z-t = -9 \\ x+3y+5z-5t = 0 \\ -x+y+3z+t = 4, \end{cases}$$

$$(p)\ \begin{cases} x+y-3z = -1 \\ 2x+y-2z = 1 \\ x+y+z = 3 \\ x+2y-3z = 1, \end{cases} \quad (r)\ \begin{cases} x+2y+3z+t = 1 \\ 2x+4y-z+2t = 2 \\ 3x+6y+10z+3t = 3 \\ x+y+z+t = 0, \end{cases} \quad (s)\ \begin{cases} x-3y+6z+4t = 5 \\ -x+2y-4z-3t = -4 \\ x+y-2z = 1 \\ 2x+3y-6z-t = 1, \end{cases}$$

$$(t)\ \begin{cases} 2t-x-y = z-2 \\ 3+y = 3t+z \\ z+3t = 2x \\ 2t+x = y-z+4, \end{cases} \quad (u)\ \begin{cases} x+y+z+2t = 1 \\ x+3y+5z+2t = -3 \\ x+2y+3z+2t = -1 \\ 3x+2y+z+6t = 5, \end{cases} \quad (v)\ \begin{cases} x+2y-z-5t = -2 \\ 2x+y-4t = 2 \\ -2x+y+z = 1 \\ -y+2z+2t = 6. \end{cases}$$

Odpowiedzi

1

(a) $\begin{bmatrix} 5 & 5 \\ -2 & 9 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & -1 & 3 \\ -1 & 5 & 4 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 6 \\ -6 \end{bmatrix}$,

(e) $\begin{bmatrix} 0 & 0 & 14 \\ -10 & -4 & 16 \\ 6 & 4 & -10 \end{bmatrix}$, (f) $\begin{bmatrix} -7 & 2 \\ 18 & -17 \end{bmatrix}$, (g) $\begin{bmatrix} -9 \\ -15 \end{bmatrix}$, (h) $\begin{bmatrix} -4 & -6 \\ 4 & 6 \\ 3 & 1 \\ -1 & -5 \end{bmatrix}$.

2

(a) -7, (b) 3, (c) $-a^2-a+3$, (d) -20, (e) $a^3+a^2-18a-14$, (f) -21, (g) 12, (h) -71,
(i) -42, (j) -38, (k) -56, (l) $3x^2+26x+21$, (m) 23, (n) 536, (o) $-6x^3-x^2+26x$, (p) -10.

3

(a) $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, (c) nie istnieje, (d) $\begin{bmatrix} -2 & -1 \\ -3 & -2 \end{bmatrix}$, (e) $\frac{1}{12}\begin{bmatrix} -5 & 8 & 1 \\ 4 & -4 & 4 \\ 3 & 0 & -3 \end{bmatrix}$,

(f) nie istnieje, (g) $\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$, (h) $\frac{1}{6}\begin{bmatrix} -1 & -3 & 1 \\ 3 & 3 & 3 \\ 5 & 3 & 1 \end{bmatrix}$, (i) $\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 2 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}$, (j) $\frac{1}{6}\begin{bmatrix} -3 & -6 & -9 & 3 \\ 2 & 0 & 8 & -2 \\ -1 & 0 & -1 & 1 \\ 6 & 6 & -6 & 0 \end{bmatrix}$.

4

(a) $\begin{bmatrix} -\frac{13}{2} & \frac{15}{2} \\ 5 & -6 \end{bmatrix}$, (b) $\begin{bmatrix} -\frac{1}{2} & 0 & \frac{3}{2} \\ -2 & 3 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}$, (c) $\begin{bmatrix} 1 & -\frac{3}{2} & 4 \\ 2 & 6 & -6 \\ 2 & \frac{11}{2} & -5 \end{bmatrix}$, (d) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & -2 \end{bmatrix}$, (e) brak rozwiązania,

(f) $\frac{1}{2}\begin{bmatrix} -6 & 2 & 4 \\ -3 & 3 & 1 \end{bmatrix}$, (g) $\frac{1}{2}\begin{bmatrix} 4 & 0 & -6 \\ 3 & -1 & -3 \end{bmatrix}$, (h) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$, (i) $\frac{3}{2}\begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & 5 \end{bmatrix}$.

5

(a) $x=-19, y=-8$, (b) $x=\frac{1}{2}, y=0$, (c) $x=2, y=-2, z=3$,
(d) $x=-7, y=3, z=6$, (e) $x=-\frac{1}{5}, y=-\frac{2}{5}, z=\frac{3}{5}$, (f) $x=\frac{1}{5}, y=-\frac{2}{5}, z=\frac{4}{5}$,
(g) $t=-\frac{7}{5}, u=\frac{4}{5}, w=-\frac{8}{5}$, (h) $x=0, y=-\frac{1}{4}, z=-\frac{3}{8}$, (i) $x=1, y=0, z=0, t=0$.

6

(a) $x=-t+2, y=t+2z-1, z,t\in\mathbb{R}$, (b) $x=2t-z+1, y=-t+3z, z,t\in\mathbb{R}$, (c) $x=2-t, y=-1, z=2t, t\in\mathbb{R}$,
(d) $x=-1-t, y=-2, z=-t, t\in\mathbb{R}$, (e) układ jest sprzeczny, (f) $x=2t+3, y=t-1, z=3,\ t\in\mathbb{R}$,
(g) $x=t, y=-3t-2, z=t+1,\ t\in\mathbb{R}$, (h) $x=t-u, y=2t+u+1-2, z=2-u,\ t,u\in\mathbb{R}$, (i) $x=1-5t, y=1-12t, z=1-17t, t\in\mathbb{R}$,
(j) układ jest sprzeczny, (k) $x=2t-z+1, y=-t+3z,\ t,z\in\mathbb{R}$, (l) $x=z-t+2, y=2z+1, z,t\in\mathbb{R}$,
(m) $x=1, y=t+z-2, z,t\in\mathbb{R}$, (n) układ jest sprzeczny, (o) $x=2t+z-3, y=t-2z+1,\ t,z\in\mathbb{R}$,
(p) układ jest sprzeczny, (r) $x=-1-t, y=1, z=0,\ t\in\mathbb{R}$, (s) $x=-t+2, y=t+2z-1,\ z,t\in\mathbb{R}$,
(t) $x=1+t, y=2t-1, z=2-t, t\in\mathbb{R}$, (u) $x=3+z-2t, y=-2-2z, z,t\in\mathbb{R}$, (v) $x=1+t, y=2t, z=3, t\in\mathbb{R}$.