**Abstract**

In this article, we present Poisson wavelets on the sphere, introduced originally by Holschneider *et al.* in [24], and provide several new results concerning them (explicit formulae for wavelets and their spatial gradient, relations between wavelets and the gradient, explicit expressions for the Euklidean limit and an algorithm for computing it, as well as some statements about the localization of wavelets and their gradient.) A prove of the existence of discrete weighted frames of Poisson wavelets, as well as a description of grids of sampling points for a frame, are given.

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# Chapter 1

# Introduction

Observations of magnetic and gravitational fields of the Earth are of great importance for the understanding of processes taking place on our planet. Therefore, a good mathematical tool is needed for the description of those fields. Until recently, the most broadly used technique has been the spherical harmonic analysis, however, it has some important disadvantages. One of them is a poor localization, such that coefficients obtained in one observation region have big influence on the coefficients in another region. On the other hand, it is difficult to distinguish the big–scale field component from the core and the small–scale field component from the crust. Further, changing the truncation level of spherical harmonics changes all the coefficients, according to the spatial aliasing of the higher–order harmonics.

Therefore, other methods have been investigated during the last years. One of them is the wavelet technique, which seems to be very promising. In this paper, we present the method developed by Holschneider *et al.* (compare [24] and [5]) using some special spherical wavelets, Poisson wavelets, constructed according to the definition given by Holschneider in [23]. Our most important result is the prove of the existence of discrete frames of Poisson wavelets and a description of the grid for sampling the wavelet coefficients. Further, we provide some new formulae concerning the wavelets.

The outline of the article is as follows. In Chapter 2 we present some introducing informations about frames and geometry of the sphere. A review of different definitions of spherical wavelets and a description of Holschneider's spherical wavelets follow (Chapter 3.) The next Chapter 4 deals with Poisson wavelets, first the definition from [24] is presented, then our original contributions to analytical representation and properties of the wavelets follow. In the final Chapter 5, we construct semicontinuous frames and find formulae for the frame bounds, further we discretize the continuous parameter such that a countable weighted frame is given.

# Chapter 2

# Preliminaries

## 2.1 A few words about frames

The concept of frames was introduced in the aricle [11] by Duffin ans Schaeffer. A frame may be understood as a generalization of a base, i.e., it is a complete set of vectors, but some of them may be redundant. In this section, we would like to give a short introduction to the theory of frames. The basic sources are [7], [**?**] and [22].

### Definition of a frame

Suppose, a countable family of vectors $\{f_n\}_{n\in\mathbb{N}}$ in a separable Hilbert space $\mathcal{H}$ is given. The question is, what features it should have so that we are able to reconstruct any vector $f \in \mathcal{H}$ from the set of its scalar products

$$a_n := \langle f_n, f\rangle_{\mathcal{H}}\,, \quad n \in \mathbb{N}.$$

Obviously, it is sufficient that $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis, but this condition is very strong. In most cases it is easier to reconstruct a signal that is oversampled (i.e., one has too much information) than to construct a basis. This consideration leads to the concept of frames.

**Definition 1** *A family of vectors* $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ *is called a* frame *for* $\mathcal{H}$ *if there exist constants* $A$, $B > 0$ *such that*

$$A\,\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\,\langle f_n, f\rangle\,|^2 \leq B\,\|f\|^2. \tag{2.1}$$

*The numbers* $A$ *and* $B$ *are called* frame bounds *(or* constants.*) The biggest possible* $A$ *and the smallest possible* $B$ *are called* optimal frame bounds. *If the optimal frame bounds are equal to each other, we say the frame is* tight. *It is* exact *if* $\{f_n\}_{\substack{n\in\mathbb{N}\\ n\neq j}}$ *is not a frame for any* $j \in \mathbb{N}$.

For better understanding of the definition, some examples are helpful. Let $M := \{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Then $M$ is a tight frame with both frame constants equal to 1. The family $\{e_1, e_1, e_2, e_2, e_3, e_3, \dots\}$ is also a tight frame with frame bounds equal to 2. Further,

$$\left\{e_1, \frac{1}{\sqrt{2}}\, e_2, \frac{1}{\sqrt{2}}\, e_2, \frac{1}{\sqrt{3}}\, e_3, \frac{1}{\sqrt{3}}\, e_3, \frac{1}{\sqrt{3}}\, e_3, \dots\right\}$$

is a frame with constants $A = 1$ and $B = 1$, but $\{(1/\sqrt{n})\, e_n\}_{n\in\mathbb{N}}$ is not a frame. A frame $\{f_n\}_{n\in\mathbb{N}}$ with its frame bounds both equal to 1 and $\|g_n\| = 1$ for $n \in \mathbb{N}$ is actually an orthonormal base.

## Bessel sequences and the frame opearator

Closely related to frames are the so–called Bessel sequences: families of vectors that satisfy only the upper bound condtion.

**Definition 2** *A set of elements* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *is called a* Bessel sequence *if there exists a constant* $B > 0$ *such that*

$$\sum_{n\in\mathbb{N}} |\,\langle f_n, f\rangle\,|^2 \le B\,\|f\|^2 \quad \forall f \in \mathcal{H}.$$

$B$ *is called a* Bessel bound.

An example for a Bessel sequence that is not a frame is the family $\{(1/\sqrt{n})\, e_n\}_{n\in\mathbb{N}}$ with $e_n$ – elements of an orthonormal basis (to see that, choose $f = e_n$.)

For a Bessel sequence, the operator

$$T : \ell^2(\mathbb{N}) \to \mathcal{H},\ \{a_n\}_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} a_n f_n$$

is bounded with $\|T\| \le \sqrt{B}$ and the sum converges strongly and unconditionally. More exactly, there exists an $\eta \in \mathcal{H}$ such that

$$\lim_{N\to\infty} \left\| \sum_{n\in I_N} a_n f_n - \eta \right\|_{\mathcal{H}} = 0,$$

where $I_N \subset I_{N+1}$ is any growing family of finite subsets of $\mathbb{N}$ with $\bigcup_{N\in\mathbb{N}} I_N = \mathbb{N}$. The operator $T$ is called the *preframe operator*. Its adjoint is given by

$$T^* : \mathcal{H} \to \ell^2(\mathbb{N}),\ f \mapsto \{\langle f_n, f\rangle\}_{n\in\mathbb{N}}.$$

It is an imperfect sampling operator. It is also bounded, by the Bessel condition.

We may thus combine sampling and reconstruction. However, in order to do it a bit more generally, we take two different Bessel sequences, that means, we consider the operator $S = T_{\{g_n\}} T^*_{\{f_n\}}$. Explicitly, we have

$$S : \mathcal{H} \to \mathcal{H},\ f \mapsto \sum_{n\in\mathbb{N}} \langle f_n, f\rangle\, g_n,$$

and we want this operator to be invertible. The following definition is helpful:

**Definition 3** *Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be Bessel sequences in $\mathcal{H}$. If there exist constants $c \in \mathbb{C}$ and $\epsilon < 1$ such that*

$$\left\| c \sum_{n\in\mathbb{N}} \langle f_n, f\rangle\, g_n - f \right\|^2_{\mathcal{H}} \le \epsilon\, \|f\|^2_{\mathcal{H}} \qquad \forall f \in \mathcal{H},$$

*then $\{f_n\}$ and $\{g_n\}$ are called a* sampling–reconstruction pair*. $\{g_n\}_{n\in\mathbb{N}}$ is a* reconstructing family *for $\{f_n\}$.*

An equivalent characterization of sampling–reconstruction pairs is to require that for all $f, g \in \mathcal{H}$ the following holds:

$$\left| c \sum_{n\in\mathbb{N}} \langle f_n, f\rangle \langle g, g_n\rangle - \langle g, f\rangle \right| \le \sqrt{\epsilon}\, \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}. \tag{2.2}$$

Therefore, if $\{g_n\}$ is a reconstructing family for $\{f_n\}$, then $\{f_n\}$ is also a reconstructing family for $\{g_n\}$, with the same constants. In that case, the theorem of Neumann applies, and the operator $S$ is invertible. Its inverse can be computed recurrently (via Neumann series.) With a little modification we obtain the following algorithm to invert the sampling operator $T^*_{\{f_n\}}$:
Given the samples $v = v_0 = \{a_n\}_{n\in\mathbb{N}}$ of a signal $\phi \in \mathcal{H}$, construct recurrently

$$\begin{aligned} \phi_k &= c\, T v_k = c \sum_{n\in\mathbb{N}} v_k(n)\, g_n, \\ v_{k+1}(n) &= (T^* \phi_k)(n) - v_k(n) = \langle f_n, \phi_k\rangle - v_k(n),\ n \in \mathbb{N}, \end{aligned}$$

for $k \in \mathbb{N}_0$. Then, the sequence $\phi_k$ converges to $\phi$ in the topology of $\mathcal{H}$.

## Relation between frames and Bessel sequences. Frame decomposition theorem

Now, the relation between Bessel sequences and frames may be stated.

**Theorem 1** *The class of Bessel sequences having a reconstruction Bessel sequence coincides with the class of frames.*

If the sequences $\{f_n\}$ and $\{g_n\}$ are a sampling–reconstruction pair, we say that $\{g_n\}$ is a *bi-frame* to $\{f_n\}$.

In the case of sampling and reconstructing with respect to the same frame $\{f_n\}$, we call the operator $S$ the *frame operator*. Then, a possible choice of the constants $c$ and $\epsilon$ in the inequality (2.2) is

$$c = \frac{2}{A+B} \quad \text{and} \quad \epsilon = \frac{B-A}{B+A}, \tag{2.3}$$

where $A$ and $B$ are the frame bounds. For $A$ and $B$ such that $A \leq 1 \leq B$ and $B - A < 2$ one can also take

$$c = 1 \quad \text{and} \quad \epsilon = \frac{B-A}{2}.$$

The frame operator is self–adjoint, positive and bounded by $B$. Its inverse satisfies

$$B^{-1}\mathbf{1} \leq S^{-1} \leq A^{-1}\mathbf{1},$$

where $U_1 \leq U_2$ means $\langle U_1 x, x\rangle \leq \langle U_2 x, x\rangle$, $\forall x \in \mathcal{H}$, for self–adjoint operators $U_1$ and $U_2$. Further, $\{S^{-1}f_n\}$ is a frame with bounds $B^{-1}$, $A^{-1}$ and the frame operator $S^{-1}$. If the bounds $A$ and $B$ were optimal for $\{f_n\}$, then the same are the bounds $B^{-1}$ and $A^{-1}$ for $\{S^{-1}f_n\}$.

As a consequence of the invertibility of the frame operator one has the following *frame decomposition theorem*:

$$f = SS^{-1}f = \sum_{n\in\mathbb{N}} \left\langle f_n, S^{-1}f \right\rangle f_n \tag{2.4}$$

for all $f \in \mathcal{H}$. Sometimes it is written in the form

$$f = S^{-1}Sf = \sum_{n\in\mathbb{N}} \langle f_n, f\rangle \, S^{-1}f_n$$

for all $f \in \mathcal{H}$. In both cases, the series converges unconditionally for all $f \in \mathcal{H}$ (similarly as in the case of Bessel sequences.) The equation (2.4) means that all the information about the function $f \in \mathcal{H}$ is contained in the sequence $\{\left\langle f, S^{-1}f_n \right\rangle\}$ of the so–called *frame coefficients*.

## Equivalent definitions of frames

Actually, it is enough to check the frame condition on a dense subset of $\mathcal{H}$. More exactly, if for a sequence $\{f_n\} \subseteq \mathcal{H}$ there exist constants $A$, $B > 0$ such that

$$A\,\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\,\langle f_n, f\rangle\,|^2 \leq B\,\|f\|^2$$

for all $f$ in a dense subset of $\mathcal{H}$, then $\{f_n\}$ is a frame for $\mathcal{H}$ with frame bounds $A$ and $B$.

An equivalent characterization of frames can be given via the preframe operator.

**Lemma 1** *A sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$ if and only if*

$$T : \{a_n\}_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} a_n f_n$$

*is a well defined mapping from $\ell^2(\mathbb{N})$ onto $\mathcal{H}$.*

## Some modifications of the frame definition

So far we have introduced only countable frames, however, some variations of the original frame concept are in use. One of them is the continuous frame.

**Definition 4** *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{M}$ a measure space with a positive measure $\mu$. A* continuous frame *is a family of vectors $\{f_\upsilon\}_{\upsilon\in\mathcal{M}}$ that satisfies*
*a) for all $f \in \mathcal{H}$, $\upsilon \mapsto \langle f, f_\upsilon\rangle$ is a measurable function on $\mathcal{M}$;*
*b) there exist constants $A$, $B > 0$ such that*

$$A\|f\|^2 \leq \int_{\mathcal{M}} |\langle f, f_\upsilon\rangle|^2 \, d\mu(\upsilon) \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Similarly, a mixed set of indices may be used, where some of them are continuous and some of them are discrete, compare [1].

**Definition 5** *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{M}$ a measure space with a positive measure $\mu$. The family $\{f_{\upsilon,n} : \upsilon \in \mathcal{M}, n \in \mathbb{N}\}$ is called a* half–continuous frame (semicontinuous frame) with weight $\nu$ *if*
*a) for all $f \in \mathcal{H}$, $n \in \mathbb{N}$, the function $\upsilon \mapsto \langle f, f_{\upsilon,n}\rangle$ is measurable on $\mathcal{M}$;*
*b) there exist constants $A$, $B > 0$ such that*

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} \int_{\mathcal{M}} |\langle f, f_{\upsilon,n}\rangle|^2 \, d\mu(\upsilon) \cdot \nu_n \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

At last, one more definition:

**Definition 6** *A family of vectors $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is called a* frame with weight $\mu = \{\mu_n\}_{n\in\mathbb{N}}$ *if there exist constants $A$, $B > 0$ such that*

$$A\,\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\,\langle f_n, f\rangle\,|^2 \cdot \mu_n \leq B\,\|f\|^2.$$

All the considerations in this chapter can be easily adapted to this modified concepts of a frame. For our purposes, the new definitions are more convenient than the original one.

## 2.2 Sphere – basic notions

In this section we would like to state some facts about functions over a sphere and fix the notation concerned with them. More details may be found in [14], [25] or [21].

By $\Omega$ we denote the unit two-dimensional sphere in $\mathbb{R}^3$, $\Omega = \{\xi \in \mathbb{R}^3 : |\xi|^2 = 1\}$, with the measure $d\omega$ invariant under rotation group and such that $\int_\Omega d\omega = 4\pi$. The unit vector in direction of the north–pole is denoted by $\hat{e}$; further, $\theta \in [0, \pi]$ is the polar (colatitudinal) coordinate, and $\phi \in [0, 2\pi)$ – the azimuthal (longitudinal) coordinate.
Zonal functions are those that depend only on $\theta$. They will be identified with functions over the interval $[-1, 1]$ and if no confusion can occur, we shall use the same symbol:

$$f(x) = f(\theta) = f(x \cdot \hat{e}) = f(\cos\theta),$$

where $x \cdot y$ denotes the scalar product between the unit vectors $x$ and $y$ having their origin in 0 and endpoints on the sphere; such vectors will be occasionally identified with points $x$ and $y$ on the sphere. By a translation of a zonal function $f$ to the point $y \in \Omega$ we mean the function $\tau_y f(x) = f_y(x) = f(x \cdot y)$. Since $f$ is rotation invariant around the $\hat{e}$–axis, its translation is rotation invariant with respect to the axis through $y$. The convolution of a zonal $f$ with an arbitrary function $g$ is defined as

$$f * g(x) = \langle \bar{f}_x, g \rangle = \int_\Omega f(y \cdot x)\, g(y)\, d\omega(y).$$

The spherical harmonics $Y_l^m$, $l \in \mathbb{N}_0$, $m = -l, -l+1, \ldots, l$, are given by

$$\begin{aligned} Y_l^m(\theta, \phi) &= (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m(cos\theta)\, e^{im\phi}, \\ Y_l^{-m} &= (-1)^m \overline{Y_l^m}, \end{aligned}$$

for $0 \leq m \leq l$, where $P_l^m$ are the (associated) Legendre polynomials. They build an orthonormal basis for $\mathcal{L}^2(\Omega)$–functions. The spherical harmonics $Y_l^0$ are obviously zonal. The linear span of spherical harmonics of the same order, $\text{span}\{Y_l^m : -l \leq m \leq l\}$, is denoted by $\Sigma_l$. The space $\mathcal{L}^2(\Omega)$ can then be decomposed into orthogonal components:

$$\mathcal{L}^2(\Omega) = \bigoplus_{l=0}^{\infty} \Sigma_l.$$

The functions

$$Q_l = \sqrt{\frac{2l+1}{4\pi}}\, Y_l^0 = \frac{2l+1}{4\pi}\, P_l$$

are reproducing kernels for the spaces $\Sigma_l$, i.e., the convolution with $Q_l$ acts on $\Sigma_l$ like the identity:

$$s \in \Sigma_k \Rightarrow Q_l * s = \delta_{k,l}\, s \quad \text{(Funk–Hecke formula.)}$$

This is an important difference to $\mathcal{L}^2(\mathbb{R})$, where the only reproducing kernel, $\delta$, is not a smooth function. The orthogonal projection $\mathbb{P}_l$ of $s \in \mathcal{L}^2(\Omega)$ onto $\Sigma_l$ is given by the convolution with the reproducing kernel of this subspace

$$s_l := \mathbb{P}_l s = Q_l * s.$$

For an $\mathcal{L}^2(\Omega)$–function $F$, the series

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \langle F, Y_l^m \rangle_{\mathcal{L}^2(\Omega)} Y_l^m$$

is the *Fourier series* (or *orthogonal expansion*) of $F$ in terms of spherical harmonics. The constants $\widehat{F}(l,m) = \langle F, Y_l^m \rangle$ are called the *Fourier coefficients* of $F$. If $F$ is a continuous function, then the following relation holds between $F$ and its Fourier–series:

$$\lim_{\substack{h \to 0 \\ h<1}} \sum_{l=0}^{\infty} h^l \sum_{m=-l}^{m=l} \widehat{F}(l,m) Y_l^m(\xi) = F(\xi),$$

and the convergence is uniform with respect to $\xi \in \Omega$ for fixed $h \in (0, h_0)$, $h_0 < 1$. The system $\{Y_l^m\}_{\substack{l \in \mathbb{N}_0 \\ -l \le m \le l}}$ is closed in $\mathcal{C}(\Omega)$ with respect to the $\mathcal{C}(\Omega)$–norm and with respect to the $\mathcal{L}^2(\Omega)$–norm, i.e., for any $F \in \mathcal{C}(\Omega)$ and any $\epsilon > 0$ there exists a linear combination

$$\sum_{l=0}^{L} \sum_{m=-l}^{m=l} a_{l,m} Y_l^m$$

such that

$$\left\| F - \sum_{l=0}^{L} \sum_{m=-l}^{m=l} a_{l,m} Y_l^m \right\| < \epsilon,$$

where $\|\cdot\|$ is any of the two norms. Analogously, $\{Y_l^m\}_{\substack{l \in \mathbb{N}_0 \\ -l \le m \le l}}$ is closed in $\mathcal{L}^2(\Omega)$ with respect to the $\mathcal{L}^2(\Omega)$–norm. Further, the Parseval identity

$$\|F\|^2_{\mathcal{L}^2(\Omega)} = \langle F, F \rangle_{\mathcal{L}^2(\Omega)} = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left| \langle F, Y_l^m \rangle_{\mathcal{L}^2(\Omega)} \right|^2 \tag{2.5}$$

and the extended Parseval identity

$$\langle F, G \rangle_{\mathcal{L}^2(\Omega)} = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \langle F, Y_l^m \rangle_{\mathcal{L}^2(\Omega)} \langle G, Y_l^m \rangle_{\mathcal{L}^2(\Omega)}$$

hold for $\mathcal{L}^2(\Omega)$–functions $F$ and $G$.

In our further considerations we will need some facts about Legendre functions. They are solutions to the Legendre differential equation:

$$(1-x^2)\frac{d^2}{dx^2}f(x) - 2x\frac{d}{dx}f(x) + \left(l\,(l+1) - \frac{m^2}{1-x^2}\right) f(x) = 0$$

for $0 \le m \le l$ on the interval $[-1,1]$, and they are given by

$$\begin{aligned} {P_l}^0(x) &= P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l && \text{(Legendre polynomials)} \\ P_l^m(x) &= (1-x^2)P_l^{(m)}(x) \quad \text{for } 0 < m \le l && \text{(associated Legendre functions.)} \end{aligned}$$

Their $\mathcal{L}^2([-1,1])$–norms equal

$$\|P_l^m\| = \sqrt{\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}}.$$

The Legendre polynomials can be represented in the integral form, as so-called *Schläfli integrals*:

$$P_l(x) = \frac{1}{2^{l+1}\pi i}\oint_{\mathcal{C}} \frac{(\zeta^2-1)^l}{(\zeta-x)^{l+1}}\,d\zeta, \tag{2.6}$$

where $\mathcal{C}$ is a closed path around $x = x + 0 \cdot i$ (actually, this formula defines a Legendre polynomial for any complex argument.) They satisfy the following *generating function equation*:

$$\sum_{l=0}^{\infty} P_l(x)\,h^l = \frac{1}{\sqrt{1+h^2-2hx}} \tag{2.7}$$

for any $h \in (-1,1)$ and $x \in [-1,1]$. For the derivative of a Legendre polynomial we have the following recurrence relation:

$$(x^2-1)\,P_l'(x) = l\,[xP_l(x) - P_{l-1}(x)]. \tag{2.8}$$

# Chapter 3

# Wavelets on the sphere

## 3.1 Review of definitions of spherical wavelets

In the last decade there were many attempts to define a wavelet transform on closed manifolds. Experimental data are very often given on the sphere, e.g., the geographical or statistical data, data in computer graphics or medical imaging. The advantage of a wavelet transform over the standard methods basing on the Fourier transform is that a local change of the signal requires only a local change in the coefficients. However, many difficulties occure when transfering the global concepts of classical wavelet theory, e.g., on $\mathbb{R}^2$, to other manifolds. Here we present an overview of some definitions. It does not attempt to be complete or exhaustive.

One of the first definitions of discrete wavelets on the sphere was given by Schröder and Sweldens in the articles [29] and [30]. It bases on a quasi–uniform icosahedral trinagulation of the sphere. On the triangles one constructs simple wavelets (like the Haar wavelet) and then obtains smoother functions by lifting. The so defined wavelets are shown to be very useful in numerical experiments. It was proven that they build a stable $\mathcal{L}^2$—basis. The disadvantage is that the scaling functions can be evaluated exactly only at the grid points.

Another definition is due to Dahlke *et al.* ([9].) Once the sphere is equiped with polar coordinates, one constructs the wavelets as tensor products of exponential splines (in the first coordinate) and interval wavelets (in the second coodinate.) The wavelets are of the class $\mathcal{C}^1$ and an MRA is given. The problem is that the construction is based on a fixed chart for the sphere; further, there are some problems at the poles when projecting functions onto the wavelet spaces. This approach is extended to stable biorthogonal spherical wavelets in Weinreich's works [31] and [32].

Similarly, a dicrete wavelet transform basing on splines is described in [18], an article by Göttelmann. Here again one has an MRA of $\mathcal{L}^2(\Omega)$. A stable wavelet basis of Sobolev spaces $H^s(\Omega)$ for $0 \leq s < 3/2$ is obtained, however, it is nesessary to modify the grid (in contrary to the simplier approach from [32].)

The poles are not exeptional points in this approach.
In the article [16] by Freeden and Schreiner radially symmetric spherical wavelets with arbitrary smoothness are defined with help of spherical harmonics. This discrete construction requires neither a fixed coordinate system nor triangulation.
Similarly, Conrad and Prestin ([8]) define a wavelet transform basing on the MRA given by spaces of spherical harmonics $\mathcal{V}_n := \bigoplus_{j=0}^{2^{n-1}-1} \Sigma_l$.

One of the attempts to define a continuous wavelet transform is described in the article [17] by Freeden and Windheuser (also described in the book [14].) Again the system of spherical harmonics is involved, one constructs a continuous MRA from wavelets with vanishing moments up to a certain order. The wavelet transform is then scale discretized in two ways: so called P–scale discretization and D–scale discretization. The D–wavelets build a (half–continuous) frame and a dual frame is constructed, whereas the P–wavelets are their own reconstruction family. A full scale–space discretized wavelet transforms are obtained by approximate integration rules on the sphere. A generalization of the continuous wavelet transform definition to Hilbert spaces is described in [15].
Another idea, performed by Rubin, is to define a continuous wavelet transform basing on the Calderòn reproducing formula ([27]) or spherical Radon transform ([28].)
A concept most similar to one that we will consider in this work is due to Antoine and Vandergheyns [3]. It is a purely group–theoretical approach basing on the construction of general coherent states of the Lorentz group $SO_0(3, 1)$. The definition is very strict and quite difficult to understand. In contrary to this, in the definition given by Holschneider [23] one has an *ad hoc* construction of the range of scales, but the approach is simplier and the wavelets have the same nice properties like, e.g., zero–mean or Euklidean limit. Another article of Antoine and Vandergheyns, [2], adapts the concept to $n$–dimensional spheres. The dilation operator introduced in this article is then generalized to a conformal one by Cerejeiras *et al.* in [4]. It allows a nice geometric description in the framework of Clifford analysis. Frames of these wavelets are constructed with use of atomic space decompositions, a concept introduced by Feichtinger and Gröchenig in [12], and adapted to the case of homogenous spaces by Dahlke *et al.* in [10].
A detailed construction of frames of wavelets by Antoine and Vandergheyns is described in [1]. Similarly as in [17], half–continuous frames are fully discretized in the next step.

## 3.2 Holschneider's definition of spherical wavelets

In this and further sections, $\mathfrak{c}$ means a constant that may change its value from expression to expression.

The definition we would like to present as the last one and the one our considerations in this thesis base on, is due to Holschneider and was first published in [23].
Since there is no natural dilation operator on the sphere, the different scales are constructed more or less artificially. One chooses a family $\{g_a\} \subset \mathcal{L}^2(\Omega)$ of wavelets indexed by a parameter $a \in \mathbb{R}_+$ and it is supposed that the mapping $a \mapsto g_a$ is continuous from $\mathbb{R}_+$ to $\mathcal{L}^2(\Omega)$. Now, the wavelet transform of a function $s \in \mathcal{L}^2(\Omega)$ is given by

$$\mathcal{W}_g s(\xi, a) = \langle U(\xi) g_a, s \rangle, \quad \xi \in SO(3),\, a \in \mathbb{R}_+, \tag{3.1}$$

where $(U(\xi)s)(x) = s(x \cdot \xi^{-1})$ (remember that $SO(3)$ is the group of rotations of $n$–dimensional space, represented by orthogonal matrices $n \times n$.) The parameter space $SO(3) \times \mathbb{R}_+$ will be denoted by $\mathbb{H}$.
This wavelet transform is in some aspects very similar to that over a plane. This makes it most suitable in many applications. Some characteristic properties are:

- The Euklidean limit property holds for the wavelets, i.e., there exists a function $g \in \mathcal{L}^2(\mathbb{R}^2)$ such that

  $$\lim_{a \to 0} a^2 g_a \left( \Phi^{-1}(a\xi) \right) = g(\xi) \quad \text{a.e.}$$

  Here, $\Phi : \Omega \backslash \{(0,0,-1)\} \to \mathbb{R}^2$ is the stereographic projection of the sphere with the south–pole removed onto the open plane (Fig. 4.4 on page 37.) This means that the scaling behaviour of $\{g_a\}$ for small scales is the same as that of the wavelets over $\mathbb{R}^2$, i.e., the sphere is locally flat and $a$ behaves asymptotically like a dilation parameter.

- For big scales the wavelet transform decays faster than any polynomial: for all $\alpha$ there exists a finite constant $c_\alpha$ such that

  $$|\mathcal{W}_g s(\xi, a)| \leq c_\alpha a^{-\alpha} \quad \forall\, a > 1.$$

  This is an equivalent of a property of the wavelet transform over a circle: since the sphere is compact, a function over it cannot have big-scale details.

- The energy conservation holds: there is a finite constant $c_g$ such that for all $s \in \mathcal{L}^2(\Omega)$ with $\int s = 0$ the following is true:

  $$\int_{\mathbb{H}} |\mathcal{W}_g s(\xi, a)|^2 \, \frac{d\sigma(\xi)\, da}{a} = c_g \int_\Omega |s(x)|^2 \, d\omega(x).$$

Now, we come to the details.

**Definition 7** *A family of functions $\{g_a\}$ is called admissible if for any $a \in \mathbb{R}_+$ the function $g_a$ is of the class $\mathcal{C}^2(\Omega)$ and its Fourier coefficiens $\widehat{g_a}(l,m) = \langle g_a, Y_l^m \rangle_{\mathcal{L}^2(\Omega)}$ satisfy:*
*a) The constant*

$$c_g(l) := \frac{8\pi^2}{2l+1} \int_{\mathbb{R}_+} \sum_{|m| \leq l} |\widehat{g_a}(l,m)|^2 \, \frac{da}{a} \tag{3.2}$$

*is finite and independent of* $l$.
*b) For* $a \to \infty$ *the sum*

$$\sum_{l=0}^{\infty} \sum_{|m| \leq l} |\widehat{g_a}(l,m)|^2$$

*decreases faster than any polynomial* ($\leq \mathcal{O}(a^{-n})$ $\forall n \in \mathbb{N}$.)
*c) There exists a function* $\gamma \in \mathcal{L}^2(\mathbb{R}_+ \times \mathbb{Z})$*, piecewise smooth in the first variable, with support (essentially) far from* $\{0\} \times \mathbb{Z}$,

$$|\gamma(t,m)| \approx 0 \quad \text{for } t < c,\ c > 0,\ m \in \mathbb{Z},$$

*and such that*

$$\lim_{a \to 0} \sqrt{l} \cdot \gamma(al,m) = \widehat{g_a}(l,m)$$

*pointwise for* $l \in \mathbb{N}_0$, $|m| \leq l$.

For an admissible wavelet family, we define the spherical wavelet transform via (3.1). As shown in [23], condition a) of Definition 7 is necessary and sufficient for the wavelet transform to satisfy the energy conservation. Analogously, large scale decay of the Fourier coefficients of the wavelets (condition b)) is sufficient for the large scale decay of the wavelet transform. Finally, condition c) ensures that the Euklidean limit property holds. We would like to prove the last statement, since there are some inaccuracies in the proof given by Holschneider in [23]. Since we consider the behaviour of the wavelets in the limit $a \to 0$, it is enough to take into account wavelet families given by

$$\widehat{g_a}(l,m) = \sqrt{l} \cdot \gamma(al,m),$$

with $\gamma$ like in Definition 7 c).

**Lemma 2** *For the following family of* $\mathcal{L}^2(\Omega)$*-functions:*

$$g_a = \sum_{l=0}^{\infty} \sum_{|m| \leq l} \widehat{g_a}(l,m)\, Y_l^m, \quad \widehat{g_a}(l,m) = \sqrt{l} \cdot \gamma(al,m), \tag{3.3}$$

*where* $\gamma$ *is an* $\mathcal{L}^2(\mathbb{R}_+ \times \mathbb{Z})$*–function, piecewise smooth in the first argument, with support (essentially) far from* $\{0\} \times \mathbb{Z}$,

$$|\gamma(t,m)| \approx 0 \quad \text{for } t < c,\ c > 0,\ m \in \mathbb{Z},$$

*and such that* $(t,m) \mapsto t\,\gamma(t,m)$ *is in* $\mathcal{L}^1(\mathbb{R}_+ \times \mathbb{Z})$*, the Euklidean limit property holds.*

*Proof.* The following holds uniformly for $\theta \in I$, $I$ compact:

$$\lim_{l \to \infty} \sqrt{2\pi/l} \cdot Y_l^m(\theta/l, \psi) = J_m(\theta)\, e^{im\psi}, \tag{3.4}$$

where
$$J_m \colon t \mapsto \frac{1}{2\pi} \int_0^{2\pi} e^{-i(m\phi - t\sin\phi)}\, d\phi$$
is the $m$-th order Bessel function, compare [26, p. 93] with $P_l^{-m}$ defined as $(-1)^m \frac{(l-m)!}{(l+m)!} P_l^m$, see also [19, p. 953, formula 8.722.2], resulting from the previous equation. Now, consider $a^2\, g_a\left(\Phi^{-1}(a\xi)\right)$ in the limit $a \to 0$. The variable $\xi \in \mathbb{R}^2$ can be written as $(\rho, \psi)$ in polar coordinates and then for small $a$ we have $\Phi^{-1}(a\xi) \approx (a\rho, \psi)$. More exactly,
$$\lim_{a\to 0} \frac{\angle(\Phi^{-1}(a\xi), \hat{e})}{a} = \lim_{a\to 0} \frac{2\arctan(a\rho/2)}{a} = \rho \tag{3.5}$$
pointwise. Using this asymptotics we can write
$$a^2\, g_a\left(\Phi^{-1}(a\xi)\right) \approx a^2 \sum_{l=0}^{\infty} \sum_{|m|\le l} l\, \gamma(al, m)\, \frac{1}{\sqrt{l}}\, Y_l^m(a\rho, \psi). \tag{3.6}$$
For spherical harmonics the relation
$$\sum_{|m|\le l} |Y_l^m|^2 = \frac{2l+1}{4\pi}$$
holds and therefore we obtain by the Schwartz inequality
$$\sum_{|m|\le l} l\, \gamma(al, m)\, \frac{1}{\sqrt{l}}\, Y_l^m(a\rho, \psi) \le \sqrt{\sum_{|m|\le l} l^2\, |\gamma(al, m)|^2} \cdot \mathfrak{c}.$$
Now, remember that $\gamma$ was small for small first argument. This property we may state precisely as
$$\sum_{l=0}^{[c/a]} \sqrt{\sum_{|m|\le l} l^2\, |\gamma(al, m)|^2} \cdot \sqrt{\frac{2 + 1/l}{4\pi}} < \epsilon$$
for some $\epsilon \ll 1$ and $a \le a_0 < 1$. Hence, in the limit $a \to 0$ the $l$'s contributing to the sum in (3.6) get large since we can neglect terms with $al < c$. Then we can use the asymptotics (3.4) for $Y_l^m$ and obtain
$$\lim_{a\to 0} a^2 g_a\left(\Phi^{-1}(a\xi)\right) = \frac{1}{\sqrt{2\pi}} \lim_{a\to 0} a \sum_{l=0}^{\infty} \sum_{|m|\le l} (al)\, \gamma(al, m)\, J_m(al\rho)\, e^{im\psi} \tag{3.7}$$
pointwise. Since $(t, m) \mapsto t\, \gamma(t, m)$ is in $\mathcal{L}^1$ and piecewise continuous in $t$, and $|J_m(al\rho)\, e^{im\psi}| \le 1$, the series
$$\sum_{l=0}^{\infty} \sum_{|m|\le l} a\, (al)\, \gamma(al, m)\, J_m(al\rho)\, e^{im\psi}$$

converges absolutely. We may therefore exchange the order of summation, as well as the order of summation and limiting, and obtain

$$\lim_{a\to 0} a^2 g_a\left(\Phi^{-1}(a\xi)\right) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \lim_{a\to 0} \sum_{l=|m|}^{\infty} a\,(al)\,\gamma(al,m)\,J_m(al\rho)\,e^{im\psi}.$$

Further, for piecewise continuous $\mathcal{L}^1$–functions, the Riemmann sum converges to the Lebesgue integral, and by substituting $r$ for $al$ and $dr$ for $a$, we can write

$$\lim_{a\to 0} a^2\, g_a\left(\Phi^{-1}(a\xi)\right) = \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}_+} \gamma(r,m)\,J_m(r\rho)\,e^{im\psi}\,r\,dr. \tag{3.8}$$

It remains to show that the expression (3.8) represents an $\mathcal{L}^2(\Omega)$–function. Let $G$ be the inverse Fourier transform of $\overline{\gamma}$ with respect to the second variable:

$$G(r,\phi) = \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} \overline{\gamma(r,m)}\,e^{im\phi}, \qquad \overline{\gamma(r,m)} = \int_0^{2\pi} G(r,\phi)\,e^{-im\phi}\,d\phi. \tag{3.9}$$

Since $g_a$ in (3.3) are $\mathcal{L}^2(\Omega)$-functions, we have by the Parseval identity:

$$\sum_{l=0}^{\infty} \sum_{|m|\le l} l\,|\gamma(al,m)|^2 < \infty,$$

and hence $G$ as a function over $\mathbb{R}^2$ given in polar coordinates is also square integrable. The following relation holds between $\widehat{G}$, the two–dimensional Fourier transform of $G$, and $\gamma$:

$$\widehat{G}(\rho,\psi) = \sum_{m\in\mathbb{Z}} e^{im(\psi-\pi/2)} \int_{\mathbb{R}_+} \overline{\gamma(r,m)\,J_m(r\rho)}\,r\,dr. \tag{3.10}$$

To see that, write the two-dimensional Fourier transform in polar coordinates

$$\widehat{G}(\rho,\psi) = \int_{\mathbb{R}_+} \int_0^{2\pi} G(r,\phi)\,e^{-ir\rho\cos(\phi-\psi)}\,d\phi\,r\,dr$$

and use the representation (3.9) for $G$

$$\widehat{G}(\rho,\psi) = \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}_+} \overline{\gamma(r,m)} \int_0^{2\pi} e^{-i(r\rho\cos(\phi-\psi)-m\phi)}\,d\phi\,r\,dr.$$

Now, the inner integral goes over the whole period of $\phi$ and hence we can translate $\phi$ to $\phi-\psi+\pi/2$ without changing the integral bounds, and obtain

$$\widehat{G}(\rho,\psi) = \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} e^{im(\psi-\pi/2)} \int_{\mathbb{R}_+} \overline{\gamma(r,m)} \int_0^{2\pi} e^{-i(r\rho\sin\phi-m\phi)}\,d\phi\,r\,dr.$$

A direct calculation yields representation (3.10).
Finally, by a comparison of (3.8) and (3.10) we obtain

$$\lim_{a\to 0} a^2\, g_a\left(\Phi^{-1}(a\xi)\right) = \frac{1}{\sqrt{2\pi}}\,\overline{\widehat{G}(\rho,-\psi+\pi/2)}.$$

Since $\widehat{G}$ is an $\mathcal{L}^2(\mathbb{R}^2)$-function, this completes the proof. □

Note that in that way we cannot prove any stronger art of convergence. Since we want to obtain a formula valid for all $\xi \in \mathbb{R}^2$, we have the asymptotics (3.4) and (3.5) (and consequently (3.6)) holding only pointwise. However, in the concrete case of Poisson wavelets, we are able to prove the convergence dominated by an integrable majorant, see Section 4.4.

The proof suggests how to construct a wavelet family with the Euklidean limit equal to a prescribed function $g \in \mathcal{L}^2(\mathbb{R}^2)$: choose $G$ such that its Fourier transform $\widehat{G}$ satisfies

$$\frac{1}{\sqrt{2\pi}}\,\overline{\widehat{G}(\rho,-\psi+\pi/2)} = g(\rho,\psi)$$

(in polar coordinates.) Then let $\overline{\gamma(r,\cdot)}$ be the Fourier transform of $G(r,\cdot)$ by constant $r$. Set

$$g_a = \sum_{l=0}^{\infty}\sum_{|m|<l} \widehat{g_a}(l,m) Y_l^m,$$

where $\widehat{g_a}$ is given by

$$\widehat{g_a}(l,m) = \frac{\gamma(al,m)}{\sqrt{\frac{8\pi^2}{2l+1}\int_{\mathbb{R}_+}\sum_{|m|\le l}|\gamma(b,m)|^2\frac{db}{b}}}.$$

It can be easily seen (and is shown in [23, Theorem 11]) that this is an admissible family.

We may now come to the invertibility of the wavelet transform, i.e., to the wavelet synthesis. Formally it is defined by

$$\mathcal{M}_h\mathcal{T}(x) = \int_{\mathbb{H}} \mathcal{T}(\xi,a)\,(U(\xi)h_a)\,(x)\,\frac{d\sigma(\xi)\,da}{a}$$

where $\{h_a\} \subset \mathcal{L}^2(\Omega)$ is a wavelet family. In analogy to the wavelet analysis over $\mathbb{R}^2$, we consider admissible pairs.

**Definition 8** *We say that the families $\{g_a\}$ and $\{h_a\}$ build an admissible analysis–reconstruction pair if they satisfy conditions b) and c) of Definition 7 and in addition*
*a) there exists a finite constant $c$ such that for $c_g(l)$ and $c_h(l)$ defined by (3.2)*

*the inequality $c_g(l) < c$, respectively $c_h(l) < c$, holds;*
*b) the Fourier coefficients $\widehat{g_a}$ and $\widehat{h_a}$ satisfy*

$$\frac{8\pi^2}{2l+1}\int_0^\infty \sum_{|m|\le l} \overline{\widehat{g_a}(l,m)}\,\widehat{h_a}(l,m)\,\frac{da}{a} = 1.$$

Condition a) ensures that the wavelet transform with respect to $\{g_a\}$ is bounded,

$$\int_{\mathbb{H}} |\mathcal{W}_g s(\xi,a)|^2\,\frac{d\sigma(\xi)\,da}{a} \le c\int_\Omega |s(x)|^2\,d\omega(x),$$

and the wavelet synthesis with respect to $\{h_a\}$ is also bounded,

$$\int_\Omega |\mathcal{M}_h \mathcal{T}(x)|^2 d\omega(x) \le c\int_{\mathbb{H}} |\mathcal{T}(\xi,a)|^2\,\frac{d\sigma(\xi)\,da}{a},$$

i.e., the wavelet transform is a partial isometry. Condition b) is necessary and sufficient for the inversion formula

$$\mathcal{M}_h\mathcal{W}_g = \mathbf{1}$$

to hold, compare [23] for proofs. This shows that the family of translated and dilated wavelets $g_{\xi,a} := U(\xi)\,g_a$, $\xi\in\Omega$, $a\in\mathbb{R}_+$, builds a continuous frame for $\mathcal{L}^2(\Omega)$.
Clearly, if $\{g_a\}$ is an admissible family, it is also an admissible analysis–reconstruction pair with itself if we only normalize it such that $c_g = 1$. We also have the relation

$$\int_{\mathbb{H}} \overline{\mathcal{W}_g s(\xi,a)}\,w(\xi,a)\,\frac{d\sigma(\xi)\,da}{a} = \int_\Omega \overline{s(x)}\,\mathcal{M}_g w(x)\,d\omega(x),$$

i.e., the wavelet synthesis is the adjoint to the wavelet analysis. Consequently,

$$\langle \mathcal{W}_g s, \mathcal{W}_h w\rangle_{\mathcal{L}^2(\mathbb{H})} = \langle s, w\rangle_{\mathcal{L}^2(\Omega)}$$

holds for admissible analysis–reconstruction pairs $\{g_a\}$ and $\{h_a\}$. Further, the image of the wavelet transform may be characterized by the reproducing kernel equation, in analogy to $\mathbb{R}^2$–wavelet analysis.

**Lemma 3** *Let $\{g_a\}$ and $\{h_a\}$ be an analysis–reconstruction pair. Then $\mathcal{T}\in$ image$(\mathcal{W}_g)$ if and only if*

$$\mathcal{T}(\xi,a) = \int_{\mathbb{H}} \Pi_{g,h}(\xi\cdot\eta^{-1},a,b)\,\mathcal{T}(\rho,\alpha)\,\frac{d\sigma(\eta)\,db}{b},$$

*where*

$$\Pi_{g,h}(\xi,a,b) = \mathcal{W}_g h_b(\xi,a)$$

*is the* reproducing kernel *of the wavelet transform with respect to $\{g_a\}$ and $\{h_a\}$.*

For proof see [23].

## Spherical wavelet transform with respect to zonal wavelets

In this article we will consider some special families of zonal wavelets. For such wavelets we may adapt the definition from Section 3.2 such that the parameter space is $\Omega \times \mathbb{R}_+$. This makes the wavelet transform more clear and allows us to save memory capacity if we want to save the wavelet transform data.

**Definition 9** *A piecewise continuous function* $\gamma : \mathbb{R}_+ \to \mathbb{C}$ *is called* admissible *if*
*a) the constant*

$$c_\gamma := \int_{\mathbb{R}_+} |\gamma(t)|^2 \, \frac{dt}{t} \tag{3.11}$$

*is finite,*
*b) the sum*

$$S := \sum_{l=0}^{\infty} (2l+1) \, |\gamma(al)|^2$$

*decreases faster than any polynomial for* $a \to \infty$.
*If* $\chi$ *is also admissible, we say that* $(\gamma, \chi)$ *is an* analysis–reconstruction pair *if*

$$\int_{\mathbb{R}_+} \bar{\gamma}(t) \, \chi(t) \, \frac{dt}{t} = 1.$$

In particular, for an admissible $\gamma$ with $c_\gamma = 1$ we have that $(\gamma, \gamma)$ is an analysis–reconstruction pair.
Now, the wavelet family $\{g_a\}$, $a \in \mathbb{R}_+$, is defined by

$$g_a = \sum_{l=0}^{\infty} \gamma(al) \, Q_l.$$

Analogously for $h_a$ with $\gamma$ replaced by $\chi$.
Obviously, $g_a$ are zonal, since $Q_l$ are zonal. Finiteness of the integral in (3.11) ensures that the support of $\gamma$ is essentially far from 0, as claimed in condition c) of Definition 7. The squared $\mathcal{L}^2$–norm of $g_a$ is equal to $S/(4\pi)$ and hence its behaviour for big scales is given by condition b). For small scales we have

$$a^2 S \to 2 \int_{\mathbb{R}_+} t \, |\gamma(t)|^2 \, dt + a \int_{\mathbb{R}_+} |\gamma(t)|^2 \, dt$$

These both integrals exist, their convergence is ensured by conditions a) (for small $t$) and b) (for big $t$, since $\gamma$ must decrease faster than any polynomial.) Consequently, there exists a constant $\mathfrak{c} \in \mathbb{R}_+$ such that for some $a_0$ we have

$$\|g_a\|^2_{\mathcal{L}^2(\Omega)} < \mathfrak{c}/a^2 \quad \forall a < a_0.$$

Further, the constant $c_\gamma$ is equal to $(1/2\pi)$ times $c_g(l)$ defined by (3.2). Finiteness of this constant ensures that condition a) in Definition 7 is satisfied.

For such a family of wavelets, the definition of the wavelet transform given in Section 3.2 may be simplified, compare [24]. As a parameter space we take $SO(3)/SO(2) \times \mathbb{R}_+ \cong \Omega \times \mathbb{R}_+$ with the measure $d\omega(x)\, da/a$. From now on, we shall use the symbol $\mathbb{H}$ for this space. The spherical wavelet transform of an arbitrary function $s \in \mathcal{L}^2(\Omega)$ is defined as the following family of scalar products over the sphere:

$$\mathcal{W}_g s(x,a) = \langle g_{x,a}, s \rangle\,, \quad x \in \Omega,\, a \in \mathbb{R}_+,$$

where $g_{x,a}$ denotes the function $g_a$ translated to the position $x$. It can be also written as a family of convolutions

$$\mathcal{W}_g s(x,a) = \overline{g_a} * s(x).$$

Using the fact that $Q_l$ are reproducing kernels of $\Sigma_l$, we can conclude from this representation that for $s$ given as a Fourier series, $s = \sum_{l,m} \hat{s}(l,m)\, Y_l^m$, its wavelet transform can be written as

$$\mathcal{W}_g s(x,a) = \sum_{l=0}^{\infty} \sum_{|m|\le l} \overline{\gamma(al)}\, \hat{s}(l,m)\, Y_l^m(x). \tag{3.12}$$

It is also clear that

$$\mathcal{W}_g s(x,a) = \sum_{l=0}^{\infty} \overline{\gamma(al)}\, s_l(x)$$

with $s_l$ standing for the projection of $s$ onto $\Sigma_l$.

For a function $\mathcal{T} : \mathbb{H} \to \mathbb{C}$, the wavelet synthesis with respect to $\{h_a\}$ is given by

$$\mathcal{M}_h \mathcal{T}(x) = \int_{\mathbb{R}_+} \mathcal{T}(x,a) * h_a \frac{da}{a} = \int_{\mathbb{H}} \mathcal{T}(y,a)\, h_{y,a}(x)\, \frac{d\omega(y)\, da}{a},$$

whenever this integral makes sense. If $(\gamma, \chi)$ is an analysis–reconstruction pair, the wavelet transform may be inverted by the wavelet synthesis,

$$\mathcal{M}_h \mathcal{W}_g = \mathbf{1}.$$

This can be easily proven by using the representation (3.12) for the wavelet transform and again the fact that $Q_l$ are reproducing kernels for $\Sigma_l$. It also follows from the general statement for spherical wavelets, see page 22, however, we have to remember that in the case of zonal wavelets the inverse wavelet transform requires integration over the sphere and not over the whole group $SO(3)$ and that is why the constants $c_g(l)$ given by (3.2) and $c_\gamma$ given by (3.11) differ by the factor $2\pi$, that is the measure of $SO(2)$.

# Chapter 4

# Definition and properties of Poisson wavelets

In this chapter we would like to introduce Poisson wavelets. They were defined by Holschneider *et al.* in [24]. In the Sections 4.1 and 4.6 we cite Holschneider's definition and some further results; we have corrected the formula (4.2) and the consequent three formulae. In Sections 4.2 to 4.5 we present our original results concerning Poisson wavelets. These are: explicit expressions for Poisson wavelets and their spatial gradient, as well as a representation of the spatial gradient in terms of the wavelets; further, an algorithm for computing the Euklidean limit and explicit expressions for it; finally, some results concerning the localization of wavelets and their surface gradient.

For Poisson wavelets of order $n$ we choose $\gamma(t) = \gamma_n(t) = t^n\, e^{-t}$, i.e.,

$$g_a^n(x) = \sum_{l=0}^{\infty} (al)^n\, e^{-al}\, Q_l(x), \quad x \in \Omega. \tag{4.1}$$

## 4.1 Poisson wavelets as multipole wavelets

These wavelets may be identified with the electromagnetic field caused by multipole sources inside the unit ball that represents the Earth. To see that, consider a monopole located at point $\zeta = \lambda\hat{e}$, $0 \le \lambda < 1$, inside the ball. Its field is the solution to the Laplace equation

$$\Delta\Psi_\lambda = \delta_\zeta,$$

where $\delta_\zeta$ is the Dirac measure located at $\zeta$, and is given by

$$\Psi_\lambda(x) = \frac{1}{4\pi\,|x-\zeta|}, \quad x \in \mathbb{R}^3 \backslash \{\zeta\}.$$

For $x \in \Omega$, $x = (\theta, \phi)$ in spherical coordinates, we can write $\sqrt{1 - 2\lambda \cos\theta + \lambda^2}$ for $|x - \zeta|$, compare Figure 4.2 on page 29 and formula (4.8) on page 29. Using the generating series expansion of Legendre polynomials (2.7), we obtain the representation

$$\Psi_\lambda(x) = \frac{1}{4\pi} \sum_{l=0}^{\infty} P_l(\cos\theta) \cdot \lambda^l, \quad x \in \Omega.$$

Now, let $\Psi_\lambda^n$ be the field caused by the multipole $\mu = (\lambda \partial_\lambda)^n \delta_{\lambda \hat{e}}$. We then have

$$\Psi_\lambda^n(x) = (\lambda \partial_\lambda)^n \Psi_\lambda(x) = \frac{1}{4\pi} \sum_{l=0}^{\infty} P_l(\cos\theta)\, l^n \lambda^l.$$

By choosing $\lambda$ to be equal to $e^{-a}$ we get

$$\Psi_{e^{-a}}^n(x) = \frac{1}{4\pi}\, a^{-n} \sum_{l=0}^{\infty} (al)^n e^{-al} P_l(\cos\theta)$$

and finally

$$g_a^n = a^n \left(2\Psi_{e^{-a}}^{n+1} + \Psi_{e^{-a}}^n\right), \tag{4.2}$$

since $Q_l(x) = \frac{2l+1}{4\pi} P_l(\cos\theta)$.
Therefore, the wavelet $g_a^n$ may be seen as the restriction to the unit sphere of the field generated by a multipole located and oriented along the $\hat{e}$–axis. More precisely, $g_a^n$ may be harmonically continuated to $\mathbb{R}^3 \backslash \{e^{-a}\hat{e}\}$. This continuation, again denoted by $g_a^n$, satisfies

$$\Delta g_a^n = a^n \left(2(\lambda\partial_\lambda)^{n+1} + (\lambda\partial_\lambda)^n\right) \delta_{\lambda\hat{e}} \quad \text{and}$$
$$g_a^n(x) = a^n \left(2(\lambda\partial_\lambda)^{n+1} + (\lambda\partial_\lambda)^n\right) \frac{1}{4\pi\, |x - \lambda\hat{e}|}$$

for $\lambda = e^{-a}$. Since $(\lambda\partial_\lambda)\, f(\lambda) = -\partial_a\, f(e^{-a})$, we may also write

$$g_a^n(x) = (-1)^{n+1} a^n \left(2\partial_a^{n+1} - \partial_a^n\right) \frac{1}{4\pi\, |x - e^{-a}\hat{e}|}.$$

The infinite sum representation is

$$g_a^n(x) = \frac{a^n}{|x|} \sum_{l=0}^{\infty} \left(\frac{\lambda}{|x|}\right)^l l^n\, Q_l(\hat{x} \cdot \hat{e}), \tag{4.3}$$

for $x$ with $|x| \geq \lambda$, $x \neq \lambda\hat{e}$, and

$$g_a^n(x) = \frac{a^n}{\lambda} \sum_{l=0}^{\infty} \left(\frac{|x|}{\lambda}\right)^l l^n\, Q_l(\hat{x} \cdot \hat{e}), \tag{4.4}$$

for $x$ with $|x| < \lambda$. To obtain the last formula from the previous one, one needs to exchange the roles of $|x|$ and $\lambda$. Both may be derived in the same manner as the formula for Poisson wavelets *on* the sphere, one only needs to write $\sqrt{|x|^2 + \lambda^2 - 2\,|x|\,\lambda\cos\theta}$ for the distance between $|x|$ and $\lambda\hat{e}$ and use the generating function equation (2.7) for $1/\big(|x|\sqrt{1 + (\lambda/|x|)^2 - 2\,(\lambda/|x|)\cos\theta}\big)$ or $1/\big(\lambda\sqrt{1 + (|x|/\lambda)^2 - 2\,(|x|/\lambda)\cos\theta}\big)$, respectively. The case $|x| = \lambda$ is obtained by the continuity argument, and the sum on the right–hand–side of (4.3) converges for $x \neq \lambda\hat{e}$.

In general, spatial Poisson wavelets are defined for any $\lambda \in [0, \infty)$ and any $x \in \mathbb{R}^3 \backslash \{\lambda\hat{e}\}$, i.e., it is not required that the source of the field is inside the unit ball. The formulae (4.3) and (4.4) keep their validity also in this case.

The representation as multipole wavelets ensures that it is possible to find an exact explicit representation of the wavelets. Further, there is no need to compute spherical harmonics of high indices (as it would be in the case of wavelets by definition of Conrad and Prestin, compare Section 3.1.)

## 4.2 Explicit expressions for Poisson wavelets

We present here two methods to obtain explicit expressions for Poisson wavelets. The first one is worked out by the author and uses the infinite sum representation (4.1). Convergence aspects are concerned and an algorithm is given how to sum up the infinite series.

Another approach is due to Holschneider *et al.*, compare [24], and uses a representation of a field caused by a multipole as a finite sum of zonal spherical harmonics (or Legendre polynomials) concentrated around the multipole. However, the calculations base on the formula (4.2) that has been corrected in this article and hence we obtain slightly different representation for wavelets as in [24].

Obviously, in both cases we obtain the same formulae for the wavelets. They are collected in the Table 4.1.

In fine, we present a method that unifies this two approaches. It allows to find an iterative algorithm for calculating the Poisson wavelets.

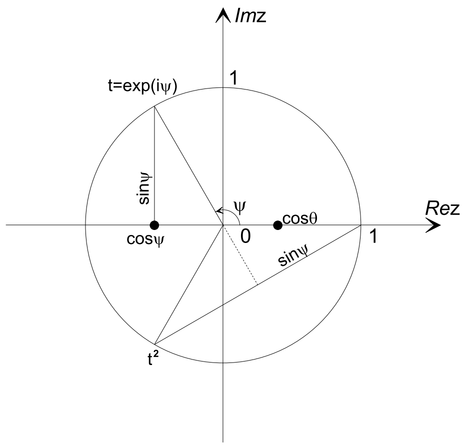
For the first approach write $x \in \Omega$ in spherical coordiantes, $x = (\theta, \phi)$, and substitute this representation in (4.1). Then, for $a \in \mathbb{R}_+$ we have

$$\begin{aligned} g_a^n(x) &= \sum_{l=0}^{\infty} (al)^n e^{-al}\,\frac{2l+1}{4\pi} P_l(\cos\theta) \\ &= \frac{2a^n}{4\pi} \sum_{l=0}^{\infty} l^{n+1}\lambda^l P_l(\cos\theta) + \frac{a^n}{4\pi} \sum_{l=0}^{\infty} l^n \lambda^l P_l(\cos\theta) \end{aligned} \tag{4.5}$$

with $\lambda = e^{-a}$. Now, take the Schläfli integral (2.6) for the Legendre polynomials:

$$P_l(\cos\theta) = \frac{1}{2^{l+1}\pi i} \oint_{\mathcal{C}} \frac{(t^2-1)^l}{(t-\cos\theta)^{l+1}}\, dt,$$

Figure 4.1: The integration path for the Schläfli integral




where $\mathcal{C}$ is a closed path that runs around the point $\cos\theta + 0 \cdot i$, and set

$$A_n = A_n(\lambda, \theta, t) = \sum_{l=0}^{\infty} l^n \frac{\lambda^l (t^2-1)^l}{2^l (t-\cos\theta)^l} = \sum_{l=0}^{\infty} l^n \alpha^l,$$

$$\text{with } \alpha = \alpha(\lambda, \theta, t) = \frac{\lambda(t^2-1)}{2(t-\cos\theta)}.$$

If only $A_n$ and $A_{n+1}$ are absolutely convergent, (4.5) can be written as

$$4\pi g_a^n(x) = 2a^n \cdot \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{1}{t-\cos\theta} A_{n+1}\, dt + a^n \cdot \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{1}{t-\cos\theta} A_n\, dt. \quad (4.6)$$

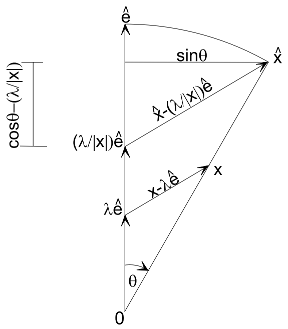
We choose $\mathcal{C}$ to be $\{t \in \mathbb{C} : t = e^{i\psi},\ \psi \in [0, 2\pi]\}$, see Figure 4.1. For $\cos\theta \in (-1,1)$ and $\lambda \in (0,1)$, the series $A_m$ converges absolutely and uniformly in $t$, since the modulus of the factor $(t^2-1)/\left(2(t-\cos\theta)\right)$ is always less than or equal to 1: for $t = e^{i\psi}$, the numerator is equal to $2\sin\psi$, and one half of the denominator is equal to the square root of $(\cos\psi - \cos\theta)^2 + (\sin\psi)^2$, hence it is bigger than or equal to $|\sin\psi|$. Consequently, for $\psi$ with $\sin\psi \neq 0$ we have

$$\left| \frac{t^2-1}{2(t-x)} \right| \leq \frac{2\sin\psi}{2\sin\psi} = 1.$$

For the other angles, $\psi = 0$ and $\psi = \pi$, the factor $(t^2-1)/\big(2(t-\cos\theta)\big)$ is equal to 0. Therefore, $|\alpha| \leq \lambda < 1$ uniformly in $t$ and we obtain the following recursive relation:

$$A_0 = \frac{\alpha}{1-\alpha}, \quad A_{m+1} = \alpha \cdot \frac{d}{d\alpha} A_m.$$

Figure 4.2: Position of the field source




The integrals in (4.6) may be computed with help of the residuum–theorem and we obtain

$$4\pi g_a^n(x) = a^n \sum_{\tau \text{ inside } \mathcal{C}} \operatorname{res}_\tau f_n \quad \text{for} \quad f_n(t) = \frac{2A_{n+1}(\lambda,\theta,t) + A_n(\lambda,\theta,t)}{t - \cos\theta}.$$

As can be easily seen, the only singular point of $f_n$ inside $\mathcal{C}$ is given by

$$z_0 = \frac{1 - \sqrt{1 + \lambda^2 - 2\lambda\cos\theta}}{\lambda},$$

and we obtain $g_a^n(x) = a^n \operatorname{res}_{z_0} f_n / 4\pi$.
For $n = 1$ we have explicitly:

$$A_1 = \frac{\alpha}{(-1+\alpha)^2} = \frac{2\lambda(-1+t^2)(t-\cos\theta)}{[-2t + \lambda(-1+t^2) + 2\cos\theta]^2}$$

$$A_2 = -\frac{\alpha(1+\alpha)}{(-1+\alpha)^3} = -\frac{2\lambda(-1+t^2)[\lambda(-1+t^2) + 2(t-\cos\theta)](t-\cos\theta)}{[-2t + \lambda(-1+t^2) + 2\cos\theta]^3}$$

$$g_a^1(x) = \frac{a}{4\pi} \cdot \frac{e^{-a}[-5e^{-a} + e^{-3a} + (3 + e^{-2a})\cos\theta]}{(1 + e^{-2a} - 2e^{-a}\cos\theta)^{5/2}}\,. \tag{4.7}$$

On the other hand we have

$$\begin{aligned} |x - \lambda\hat{e}| &= |x|^2 |\hat{x} - (\lambda/|x|)\,\hat{e}| = |x|^2 \left(\sin^2\theta + (\cos\theta - \lambda/|x|)^2\right) \\ &= |x|^2 \left(1 - (2\lambda/|x|)\cos\theta + (\lambda/|x|)^2\right) \end{aligned} \tag{4.8}$$

for $\hat{x} = x/|x|$ and $\cos\theta = \hat{x} \cdot \hat{e}$, see Figure 4.2. Formula (2.7) yields

$$g_a^1(x) = \frac{a}{4\pi} \cdot \frac{\lambda}{(1+\lambda^2-2\lambda\cos\theta)^{5/2}} [-5\lambda + \lambda^3 + (3+\lambda^2)\cos\theta]$$

$$g_a^2(x) = \frac{a^2}{4\pi} \cdot \frac{\lambda}{(1+\lambda^2-2\lambda\cos\theta)^{7/2}} [-10\lambda + 19\lambda^3 - \lambda^5 + (3 - 14\lambda^2 - 5\lambda^4)\cos\theta + (9\lambda - \lambda^3)\cos^2\theta]$$

$$g_a^3(x) = \frac{a^3}{4\pi} \cdot \frac{\lambda}{(1+\lambda^2-2\lambda\cos\theta)^{9/2}} [-20\lambda + 126\lambda^3 - 63\lambda^5 + \lambda^7 + 3(1 - 30\lambda^2 + 4\lambda^4 + 5\lambda^6)\cos\theta + 3(11\lambda - 21\lambda^3 + 6\lambda^5)\cos^2\theta + (27\lambda^2 + \lambda^4)\cos^3\theta]$$

$$g_a^4(x) = \frac{a^4}{4\pi} \cdot \frac{\lambda}{(1+\lambda^2-2\lambda\cos\theta)^{11/2}} [ - 40\lambda + 644\lambda^3 - 1008\lambda^5 + 197\lambda^7 - \lambda^9 + (3 - 394\lambda^2 + 726\lambda^4 + 246\lambda^6 - 37\lambda^8)\cos\theta + (87\lambda - 753\lambda^3 + 411\lambda^5 - 129\lambda^7)\cos^2\theta + (246\lambda^2 - 220\lambda^4 - 58\lambda^6)\cos^3\theta + (81\lambda^3 - \lambda^5)\cos^4\theta ]$$

$$g_a^5(x) = \frac{a^5}{4\pi} \cdot \frac{\lambda}{(1+\lambda^2-2\lambda\cos\theta)^{13/2}} [ - 80\lambda + 2936\lambda^3 - 10556\lambda^5 + 6616\lambda^7 - 601\lambda^9 + \lambda^{11} + (3 - 1492\lambda^2 + 8714\lambda^4 - 1626\lambda^6 - 2302\lambda^8 + 83\lambda^{10})\cos\theta$$
$$+(201\lambda - 5765\lambda^3 + 8463\lambda^5 - 3825\lambda^7 + 646\lambda^9)\cos^2\theta + (1347\lambda^2 - 5327\lambda^4 + 503\lambda^6 + 877\lambda^8)\cos^3\theta + (1554\lambda^3 - 793\lambda^5 + 179\lambda^7)\cos^4\theta + (243\lambda^4 + \lambda^6)\cos^5\theta ]$$

$$g_a^6(x) = \frac{a^6}{4\pi} \cdot \frac{\lambda}{(1+\lambda^2-2\lambda\cos\theta)^{15/2}} [ - 160\lambda + 12624\lambda^3 - 89760\lambda^5 + 126820\lambda^7 - 39090\lambda^9 + 1815\lambda^{11} - \lambda^{13}$$
$$+(3 - 5232\lambda^2 + 73170\lambda^4 - 91650\lambda^6 - 30810\lambda^8 + 14328\lambda^{10} - 177\lambda^{12})\cos\theta + (435\lambda - 35715\lambda^3 + 128805\lambda^5 - 88215\lambda^7 + 37095\lambda^9 - 2685\lambda^{11})\cos^2\theta$$
$$+(5850\lambda^2 - 68930\lambda^4 + 54600\lambda^6 + 16350\lambda^8 - 8030\lambda^{10})\cos^3\theta + (15645\lambda^3 - 34725\lambda^5 + 6480\lambda^7 - 5280\lambda^9)\cos^4\theta + (8985\lambda^4 - 2730\lambda^6 - 543\lambda^8)\cos^5\theta + (729\lambda^5 - \lambda^7)\cos^6\theta ]$$

$$g_a^7(x) = \frac{a^7}{4\pi} \cdot \frac{\lambda}{(1+\lambda^2-2\lambda\cos\theta)^{17/2}} [ - 320\lambda + 52576\lambda^3 - 677424\lambda^5 + 1822400\lambda^7 - 1278640\lambda^9 + 217230\lambda^{11} - 5459\lambda^{13} + \lambda^{15} + (3 - 17498\lambda^2 + 517002\lambda^4 - 1642530\lambda^6 + 329090\lambda^8 + 537918\lambda^{10} - 75948\lambda^{12} + 367\lambda^{14})\cos\theta$$
$$+(909\lambda - 195603\lambda^3 + 1531545\lambda^5 - 1956615\lambda^7 + 1080885\lambda^9 - 317991\lambda^{11} + 10002\lambda^{13})\cos^2\theta + (22335\lambda^2 - 664855\lambda^4 + 1457915\lambda^6 - 201435\lambda^8 - 371905\lambda^{10} + 56285\lambda^{12})\cos^3\theta$$
$$+(115230\lambda^3 - 725095\lambda^5 + 418965\lambda^7 - 147210\lambda^9 + 82610\lambda^{11})\cos^4\theta + (154440\lambda^4 - 213135\lambda^6 + 10473\lambda^8 + 29658\lambda^{10})\cos^5\theta + (49299\lambda^5 - 9299\lambda^7 + 1636\lambda^9)\cos^6\theta + (2187\lambda^6 + \lambda^8)\cos^7\theta ]$$

$$g_a^8(x) = \frac{a^8}{4\pi} \cdot \frac{\lambda}{(1+\lambda^2-2\lambda\cos\theta)^{19/2}} [ - 640\lambda + 215104\lambda^3 - 4748032\lambda^5 + 22030864\lambda^7 - 29188000\lambda^9 + 11557240\lambda^{11} - 1162576\lambda^{13} + 16393\lambda^{15} - \lambda^{17}$$
$$+(3 - 56702\lambda^2 + 3303166\lambda^4 - 21088854\lambda^6 + 21209510\lambda^8 + 7120298\lambda^{10} - 5735442\lambda^{12} + 369346\lambda^{14} - 749\lambda^{16})\cos\theta$$
$$+(1863\lambda - 988525\lambda^3 + 15351123\lambda^5 - 37427505\lambda^7 + 28089295\lambda^9 - 14071677\lambda^{11} + 2413515\lambda^{13} - 34777\lambda^{15})\cos^2\theta$$
$$+(78822\lambda^2 - 5397392\lambda^4 + 25841390\lambda^6 - 18348680\lambda^8 - 5722130\lambda^{10} + 5189072\lambda^{12} - 335162\lambda^{14})\cos^3\theta + (706605\lambda^3 - 10502545\lambda^5 + 15701510\lambda^7 - 5041350\lambda^9 + 3881315\lambda^{11} - 919615\lambda^{13})\cos^4\theta$$
$$+(1809270\lambda^4 - 6970700\lambda^6 + 2644572\lambda^8 + 684084\lambda^{10} - 756218\lambda^{12})\cos^5\theta + (1376874\lambda^5 - 1256086\lambda^7 + 89578\lambda^9 - 159742\lambda^{11})\cos^6\theta + (261804\lambda^6 - 31160\lambda^8 - 4916\lambda^{10})\cos^7\theta + (6561\lambda^7 - \lambda^9)\cos^8\theta ]$$

$$g_a^9(x) = \frac{a^9}{4\pi} \cdot \frac{\lambda}{(1+\lambda^2-2\lambda\cos\theta)^{21/2}} [ - 1280\lambda + 871296\lambda^3 - 31714752\lambda^5 + 237971328\lambda^7 - 534219504\lambda^9 + 401378880\lambda^{11} - 97176744\lambda^{13} + 6075168\lambda^{15} - 49197\lambda^{17} + \lambda^{19}$$
$$+(3 - 179760\lambda^2 + 19789206\lambda^4 - 227102526\lambda^6 + 510044430\lambda^8 - 104583822\lambda^{10} - 189309330\lambda^{12} + 50416026\lambda^{14} - 1703226\lambda^{16} + 1515\lambda^{18})\cos\theta$$
$$+(3777\lambda - 4722897\lambda^3 + 136663107\lambda^5 - 604428909\lambda^7 + 713805015\lambda^9 - 443024673\lambda^{11} + 172439043\lambda^{13} - 16686813\lambda^{15} + 115566\lambda^{17})\cos^2\theta$$
$$+(264411\lambda^2 - 39121887\lambda^4 + 363911079\lambda^6 - 587517315\lambda^8 + 92454075\lambda^{10} + 183593361\lambda^{12} - 57883497\lambda^{14} + 1792749\lambda^{16})\cos^3\theta$$
$$+(3851106\lambda^3 - 122190873\lambda^5 + 391352115\lambda^7 - 241478790\lambda^9 + 109114320\lambda^{11} - 76367319\lambda^{13} + 8284857\lambda^{15})\cos^4\theta + (16819005\lambda^4 - 147642495\lambda^6 + 154554078\lambda^8 - 13879446\lambda^{10} - 34710081\lambda^{12} + 12813843\lambda^{14})\cos^5\theta$$
$$+(24544674\lambda^5 - 62801550\lambda^7 + 17357298\lambda^9 - 4775358\lambda^{11} + 6411720\lambda^{13})\cos^6\theta + (11470746\lambda^6 - 7190346\lambda^8 + 167946\lambda^{10} + 838038\lambda^{12})\cos^7\theta + (1361508\lambda^7 - 103341\lambda^9 + 14757\lambda^{11})\cos^8\theta + (19683\lambda^8 + \lambda^{10})\cos^9\theta ]$$

with $\lambda = e^{-a}$

Table 4.1: Poisson wavelets of order 1 to 9

$$\frac{1}{|x-\lambda\hat{e}|}=\frac{1}{|x|}\sum_{l=0}^{\infty}P_l(\cos\theta)\cdot\left(\frac{\lambda}{|x|}\right)^l$$

and therefore we obtain

$$\partial_\lambda^n\,\frac{1}{|x-\lambda\hat{e}|}\bigg|_{\lambda=0}=n!\,\frac{P_n(\hat{x}\cdot\hat{e})}{|x|^{n+1}}.$$

Consequently, for a field $\Psi_\lambda^n$ caused by multipole located at $\lambda\hat{e}$ we may write

$$4\pi\Psi_\lambda^n(x)=\partial_\lambda^n\frac{1}{|x-\lambda\hat{e}|}=n!\,\frac{P_n(\cos\chi)}{|x-\lambda\hat{e}|^{n+1}},\quad \cos\chi=\frac{x-\lambda\hat{e}}{|x-\lambda\hat{e}|}\cdot\hat{e}.$$

We put this expression into (4.2) and obtain the following representation for the wavelet $g_a^n$:

$$g_a^n(x)=\frac{a^n}{4\pi}\sum_{k=1}^{n+1}k!\,(2C_k^{n+1}+C_k^n)\,e^{-ka}\,\frac{P_k(\cos\chi)}{|x-\lambda\hat{e}|^{k+1}},$$

where the coefficients $C_k^n$ are defined through

$$(\lambda\partial_\lambda)^n=\sum_{k=1}^{n+1}C_k^n\lambda^k\partial_\lambda \text{ for } k\le n \quad\text{and}\quad C_k^n=0 \text{ for } k>n.$$

They can be computed recursively via

$$C_k^{n+1}=kC_k^n+C_{k-1}^n.$$

For $n=1$ we have explicitly

$$g_a^1(x)=\frac{a}{4\pi}\left[(2+1)\,e^{-a}\cos\chi\frac{1}{|x-e^{-1}\hat{e}|^2}+2\cdot2\cdot e^{-a}\cdot\frac{1}{2}\,(3\cos^2\chi-1)\frac{1}{|x-e^{-1}\hat{e}|^3}\right].$$

By substituting $\sqrt{1-2e^{-a}\cos\theta+e^{-2a}}$ for $|x-e^{-1}\hat{e}|$ and $\frac{\cos\theta-e^{-a}}{\sqrt{1-2e^{-a}\cos\theta+e^{-2a}}}$ for $\cos\chi$ we obtain the same expression as in (4.7).

Note that in both cases we derive an infinite sum $\sum_{l=0}^{\infty}l^n\mathcal{Y}^l$ and multiply it by $\mathcal{Y}$ (whatever it is) in order to obtain some higher powers of $l$. In the first case we do it explicitly, whereas by the second method this work have been done in Section 4.1. Actually both methods are equivalent, but in the second approach more physical meaning is given to the wavelets. By the first method, summing up the integrands in Schläfli integral, we may derive the generating function equation needed for the multipole interpretation of the wavelets from Section 4.1. This remark leads us to the conclusion that actually only the first–order wavelet is needed to be calculated by some (more or less) sophistacated method, higher–order wavelets may be obtained recursivly via

$$g_a^{n+1}(x)=a\lambda\,\frac{\partial}{\partial\lambda}\,g_a^n(x), \tag{4.9}$$

where $g_a^n(x)$ is written in the form $\sum_{l=0}^{\infty}(al)^n\lambda^l\,Q_l(x)$. Another simple relation between wavelets of order $n$ and $n+1$ is

$$g_a^{n+1}(x) = -a^{n+1}\,\frac{d}{da}\,\frac{g_a^n(x)}{a^n}.$$

Using the first formula, we can derive an algorithm for iterative calculation of the coefficients of the wavelets in the representation used in the Table 4.1. A short look at this table leads us to the assumption that a wavelet of order $n$ may be written as

$$g_a^n(x) = \frac{a^n}{4\pi}\,D_{2n+3}\sum_{k=0}^{n} R_k^n(\lambda)\cos^k\theta, \quad x=(\theta,\phi),\ \lambda = e^{-a}, \tag{4.10}$$

where

$$D_k = D_k(\lambda,\theta) = \frac{\lambda}{(1+\lambda^2-2\lambda\cos\theta)^{k/2}}$$

and $R_k^n$ is a polynomial of order $2n+1-k$, actually one with non–zero coefficients only by odd or only by even powers of the argument. This is obviously true for $g_a^1$ and will be proven by induction for wavelets of higher order. Suppose, (4.10) holds for some $n$. Then, by the product rule for derivative and using the relation

$$\frac{\partial}{\partial\lambda}D_k = \frac{1-(k-1)\lambda^2+(k-2)\lambda\cos\theta}{\lambda}\cdot D_{k+2}$$

we obtain

$$\frac{4\pi}{a^n}\,g_a^{n+1}(x) = D_{2n+5}\left[1-2(n+1)\lambda^2+(2n+1)\lambda\cos\theta\right]\sum_{k=0}^{n} R_k^n(\lambda)\cos^k\theta \tag{4.11}$$

$$+\,D_{2n+5}\,(1+\lambda^2-2\lambda\cos\theta)\,\lambda\sum_{k=0}^{n} R_k^{n\,\prime}(\lambda)\cos^k\theta. \tag{4.12}$$

Collecting the powers of $\cos\theta$ leads to

$$\frac{4\pi}{a^n}\,g_a^{n+1}(x) = D_{2n+5}\left[B_0+\sum_{k=1}^{n}(B_k+C_k)\cos^k\theta + C_{n+1}\cos^{n+1}\theta\right], \tag{4.13}$$

where the coefficients $B_k = B_k(\lambda)$, $k=0,\dots,n$, and $C_k = C_k(\lambda)$, $k=1,\dots,n+1$, are given by

$$B_k = \left(1-2(n+1)\lambda^2\right)R_k^n(\lambda) + (1+\lambda^2)\,\lambda\,R_k^{n\,\prime}(\lambda), \tag{4.14}$$

$$C_k = (2n+1)\,\lambda\,R_{k-1}^n(\lambda) - 2\lambda^2 R_{k-1}^{n\,\prime}(\lambda). \tag{4.15}$$

This formula applied to $g_a^1$ with $R_0^1(\lambda) = -5\lambda+\lambda^3$ and $R_1^1(\lambda) = 3+\lambda^2$ yields

$$\begin{aligned} g_a^2(x) = {} & \frac{a^2}{4\pi}\cdot\frac{\lambda}{(1+\lambda^2-2\lambda\cos\theta)^{5/2}}\big((1-4\lambda^2)(-5\lambda+\lambda^3) \\ & + (1+\lambda^2)\,\lambda\,(-5+3\lambda^2) + [(1-4\lambda^2)(3+\lambda^2) + (1+\lambda^2)\,\lambda\cdot 2\lambda \\ & + 3\lambda(-5\lambda+\lambda^3) - 2\lambda^2\,(-5+3\lambda^2)]\cos\theta + [3\lambda\,(3+\lambda^2) - 2\lambda^2\cdot 2\lambda]\cos^2\theta\big), \end{aligned}$$

which is a correct expression for $g_a^2$. As can be easily seen from the recursion (4.13), we have also proven the assumption (4.10).
Further, iterative formulae may be derived for the *coefficients* of polynomials in the representation (4.10). Suppose, $R_k^n$ is given by

$$R_k^n(\lambda) = \sum_{j=0}^{[(2n+1-k)/2]} a_j^{n,k} \lambda^{2j+(k-1)_{\mathrm{mod}\,2}}. \tag{4.16}$$

From (4.13) we obtain for an even $k$:

$$b_0^{n+1,k} = 2a_0^{n,k}; \qquad b_{n+1-k/2}^{n+1,k} = -(k+1)\, a_{n-k/2}^{n,k};$$
$$b_j^{n+1,k} = 2\,(j+1)\, a_j^{n,k} + \big(2\,(j-n)-3\big)\, a_{j-1}^{n,k},\ j=1,\dots,n-k/2;$$
$$c_j^{n+1,k} = (2n+1-4j)\, a_j^{n,k-1},\ j=1,\dots n+1-k/2.$$

For an odd $k$ we have:

$$b_0^{n+1,k} = a_0^{n,k}; \qquad b_{n+1-[k/2]}^{n+1,k} = -(k+1)\, a_{n-[k/2]}^{n,k};$$
$$b_j^{n+1,k} = -2\,(j-n-2)\, a_{j-1}^{n,k} + (2j+1)\, a_j^{n,k},\ j=1,\dots n-[k/2];$$
$$c_j^{n+1,k} = (2n+3-4j)\, a_j^{n,k-1},\ j=1,\dots,n-[k/2].$$

The coefficients of the polynomials $R_k^n$ are then given by

$$a_j^{n+1,0} = b_j^{n+1,0},\ j=0,\dots,n+1; \qquad a_j^{n+1,n+1} = c_j^{n+1,n+1},\ j=0,\dots,[(n+1)/2];$$
$$a_j^{n+1,k} = b_j^{n+1,k} + c_j^{n+1,k},\ j=0,\dots,n+1-[(k-1)/2],\ k=1,\dots,n.$$

## 4.3 Spatial gradient of Poisson wavelets

For many purposes, the spatial wavelets are needed. In order to find an explicit representation of a spatial wavelet, one picks up a formula from the Table 4.1 on page 30, substitutes $\lambda/r$ for $\lambda$ and divides the whole expression by $r$ for $x$ with $|x| \geq \lambda$ or substitues $r/\lambda$ for $\lambda$ and divides the resulting formula by $\lambda$ if $|x| < \lambda$, compare formulae (4.1), (4.3) and (4.4). Now, the gradient of this function may be computed directly. However, we propose here another method that allows a representation of the gradient in terms of wavelets.

First, we consider the harmonic continuation of Poisson wavelets outside the ball of radius $\lambda$, i.e., for $x \in \mathbb{R}^3$ with $|x| \geq \lambda$ and different from $\lambda\hat{e}$. Let $g_a^n$ be given by (4.3), that is, in spherical cordinates

$$g_a^n(x) = \frac{a^n}{4\pi r} \sum_{l=0}^{\infty} \left(\frac{\lambda}{r}\right)^l l^n (2l+1)\, P_l(\cos\theta), \quad x=(r,\theta,\phi),\ \lambda = e^{-a}.$$

For the radial component of the gradient

$$\nabla g_a^n = \frac{\partial g_a^n}{\partial r}\,\vec{e}_r + \frac{1}{r}\frac{\partial g_a^n}{\partial \theta}\,\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial g_a^n}{\partial \phi}\,\vec{e}_\phi$$

we obtain by a direct calculation (remember that the sums converge absolutely):

$$\frac{\partial g_a^n}{\partial r} = -\frac{1}{r}\left(g_a^n + \frac{1}{a}\,g_a^{n+1}\right),$$

further, since the functions are rotation invariant, we have $\partial g_a^n/\partial\phi = 0$. The polar component is given by

$$\frac{1}{r}\frac{\partial g_a^n}{\partial\theta} = \frac{a^n}{4\pi r^2}\,(2S_{n+1} + S_n),$$

where $S_n = S_n(r,\theta)$ represents the infinite sum $\sum_{l=1}^{\infty}(\lambda/r)^l l^n P_l'(\cos\theta)(-\sin\theta)$ (note that the range of summation is restricted to $\mathbb{N}$.) For $\theta \neq 0, \pi$ we may use the formula (2.8) and obtain

$$S_n = \sum_{l=1}^{\infty}\left(\frac{\lambda}{r}\right)^l l^{n+1} P_l(\cos\theta)\cot\theta - \sum_{l=1}^{\infty}\left(\frac{\lambda}{r}\right)^l l^{n+1} P_{l-1}(\cos\theta)\operatorname{cosec}\theta.$$

The sums $B_n := \sum_{l=1}^{\infty}(\lambda/r)^l l^n P_l(\cos\theta)$ represent the field caused by single multipoles inside the earth (we only need to divide that expression by $4\pi r$) and we have the relation

$$2B_{n+1} + B_n = \frac{4\pi r}{a^n}\,g_a^n, \quad n \in \mathbb{N}. \tag{4.17}$$

However, we will also need the expressions for 'single' $B$'s (at least $B_0$ and $B_1$) in order to find values of the other sums $C_n = \sum_{l=1}^{\infty}(\lambda/r)^l l^n P_{l-1}(\cos\theta)$. From the equation above we obtain the recursive relation

$$B_{n+1} = \frac{1}{2}\left(\frac{4\pi r}{a^n}\,g_a^n - B_n\right).$$

$B_0$ and $B_1$ may be calculated directly from the generating function equation:

$$B_0 = \left.\frac{1}{\sqrt{1+\lambda_r^2 - 2\lambda_r\cos\theta}}\right|_{\lambda_r=\lambda/r} = \frac{r}{\sqrt{r^2+\lambda^2-2\lambda r\cos\theta}}$$

$$B_1 = \lambda_r\,\partial_{\lambda_r}\left.\frac{1}{\sqrt{1+\lambda_r^2 - 2\lambda_r\cos\theta}}\right|_{\lambda_r=\lambda/r} = \frac{\lambda r\,(r\cos\theta-\lambda)}{(r^2+\lambda^2-2\lambda r\cos\theta)^{3/2}}.$$

Now, in order to find explicit expressions for the $C$'s, we translate the summation index $l$ and develope the term $(l+1)^n$. After exchanging the order of summation (this is possible since the inner sum is finite), we obtain

$$C_n = \frac{\lambda}{r}\cdot\sum_{k=0}^{n}\binom{n}{k}B_k$$

for $n \in \mathbb{N}$. In this sum, $B$'s may be collected such that it becomes a weighted sum of wavelets, but probably some 'single' $B_0$ and $B_1$ will remain. The formula in terms of $B$'s and $C$'s looks like

$$\frac{1}{r}\frac{\partial g_a^n}{\partial \theta} = \frac{a^n}{4\pi r^2}\left[(2B_{n+1} + B_{n+2})\cot\theta - (2C_{n+1} + C_{n+2})\operatorname{cosec}\theta\right].$$

In the end, we would like to check the results from this section for the wavelet of the lowest order $n = 1$. We have

$$\frac{\partial g_a^1}{\partial r}(x) = \frac{3a\lambda(\lambda^2 - r^2)\left[4(\lambda^2 + r^2)\cos\theta + \lambda r\left(\cos(2\theta) - 9\right)\right]}{8\pi(\lambda^2 + r^2 - 2\lambda r\cos\theta)^{7/2}}$$

for the radial derivative and for the colatitudinal one:

$$\begin{aligned}\frac{1}{r}\frac{\partial g_a^1}{\partial \theta}(x) &= \frac{a}{4\pi r^2}\left[(2B_3 + B_2)\cot\theta - (2C_3 + C_2)\operatorname{cosec}\theta\right] \\ &= \frac{a}{4\pi r^2}\left[\frac{4\pi r}{a^2}\, g_a^2(x)\cot\theta - \left(\frac{3\lambda}{\sqrt{r^2 + \lambda^2 - 2\lambda r\cos\theta}}\right.\right. \\ &\quad + \left.\left.\frac{\lambda}{r}(8B_1 + 7B_2 + 2B_3)\right)\operatorname{cosec}\theta\right].\end{aligned}$$

For the sum of $B$'s we use again the relation (4.17) and obtain

$$\begin{aligned}8B_1 + 7B_2 + 2B_3 &= 5B_1 + 3\,(2B_2 + B_1) + 2B_3 + B_2 \\ &= \frac{5\lambda r\,(r\cos\theta - \lambda)}{(r^2 + \lambda^2 - 2\lambda r\cos\theta)^{3/2}} + \frac{12\pi r}{a}\, g_a^1(x) + \frac{4\pi r}{a^2}\, g_a^2(x).\end{aligned}$$

The resulting formula is

$$\begin{aligned}\frac{1}{r}\frac{\partial g_a^1}{\partial \theta}(x) = \frac{1}{r^2}&\left[\frac{a\lambda\,(-3r^2 + 2\lambda^2 + \lambda r\cos\theta)}{4\pi\,(r^2 + \lambda^2 - 2\lambda r\cos\theta)^{3/2}}\operatorname{cosec}\theta\right. \\ &\left. - 3a\lambda g_a^1(x)\operatorname{cosec}\theta + (r\cot\theta - \lambda\operatorname{cosec}\theta)\, g_a^2(x)\ \right],\end{aligned}$$

and explicitly we obtain

$$\frac{1}{r}\frac{\partial g_a^1}{\partial \theta}(x) = -\frac{3a\lambda\left[r^4 + 2\lambda^4 - 7\lambda^2 r^2 + \lambda r(3r^2 + \lambda^2)\cos\theta\right]\sin\theta}{4\pi(r^2 + \lambda^2 - 2\lambda r\cos\theta)^{7/2}}.$$

The results coincide with that obtained by a direct differentiation.
In the other case, for $x$ with $|x| < \lambda$, only the radial derivative changes. We obtain directly

$$\frac{\partial g_a^n}{\partial r} = \frac{1}{ar}\, g_a^{n+1}.$$

## 4.4 The Euklidean limit

Having explicit formulea for $g_a^n$, we can compute the Euklidean limit for the wavelets (using repeatidly the rule of de l'Hospital.) For $n = 1$ we obtain

$$g^1(\xi) = \frac{2 - |\xi|^2}{2\pi\,(1 + |\xi|^2)^{5/2}}.$$

Figure 4.3: $a^2\, g_a^1$ for $a = 1,\ 0.5,\ 0.25,\ 0.125$

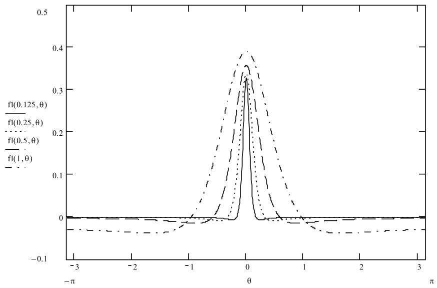



Figure 5.9 shows the function $a^2\, g_a^1$ for some different values of $a$. Note that this is (in general) the two–dimensional Fourier–transform of a function given by the formula (3.9), which is otherwise difficult to calculate. For $n = 1$ we have explicitly

$$\mathcal{F}\gamma(\omega) = (2\pi)^{3/2}\, g^1(\omega),$$

for $\gamma : \mathbb{R}^2 \to \mathbb{R}$, $\xi \mapsto |\xi|\, e^{-|\xi|}$.
For higher–order wavelets, the direct calculation of the Euklidean limit may be very difficult. In order to make it easier, one can compute $\lim_{a\to 0}[a^2\, g_a^n(\Phi^{-1}(a\xi))]^2$, since then one has to deal with a rational function. It is then not a problem to decide if the Euklidean limit of the wavelet is the positive or the negative square root of the resulting expression.

## Derivation of the formulae

However, we have noticed that the structure (4.10) of the wavelets may be exploited. We suppose, an iterative formula for the coefficients of $g^n$ may be found. Some partial results are presented below.

The colatitude $\theta$ of $\Phi^{-1}(a\xi)$, $\xi \in \mathbb{R}^2$, is given by

$$\cos\theta = \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{4 - a^2\rho^2}{4 + a^2\rho^2},$$

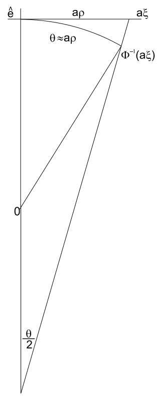
where $\rho$ is the radius of $\xi$ in polar coordinates, compare Fig. 4.4. Now, for any natural $m$ the following holds

$$\lim_{a\to 0} \frac{a^m\, e^{-a}}{\left[1 + e^{-2a} - 2e^{-a} \cdot \frac{4 - a^2\rho^2}{4 + a^2\rho^2}\right]^{m/2}} = \frac{1}{(1+\rho^2)^{m/2}} \quad \text{pointwise,} \tag{4.18}$$

and hence we only need to find the value of the limit:

$$G_n(\xi) := \lim_{a\to 0} \frac{\sum_{k=0}^{n} R_k^n(e^{-a})\, \cos^k\theta}{a^{n+1}}. \tag{4.19}$$

Figure 4.4: Stereographic projection




Further, for $a\rho < 4$ and $k \in \mathbb{N}$ we have

$$\left(\frac{4-a^2\rho^2}{4+a^2\rho^2}\right)^k = \sum_{p=0}^{\infty} c_{k,p}\,(a\rho)^{2p}$$

with some coefficients $c_{k,p}$. For $k = 0$ they are equal to $c_{0,0} = 1$ and $c_{0,p} = 0$ for $p \geq 1$, and for $k = 1$ they are given by $c_{1,0} = 1$ and $c_{1,p} = 2 \cdot (-1)^p/4^p$ for $p \geq 1$. Note that this sum converges absolutely for $a < 4/\rho$ (compare coefficients $c_{1,p}$), and hence we can exchange the order of summation and limit in (4.19):

$$G_n(\xi) = \lim_{a\to 0} \frac{\sum_{k=0}^{n} R_k^n(e^{-a})}{a^{n+1}} + \sum_{p=1}^{\infty} \lim_{a\to 0} \frac{\sum_{k=1}^{n} c_{k,p}\, R_k^n(e^{-a}) \cdot (a\rho)^{2p}}{a^{n+1}}. \tag{4.20}$$

Suppose $2p > n+1$; then the second limit is equal to 0. Therefore, the summation index $p$ may be taken from 1 to $[\frac{n+1}{2}]$. We write $\sum_{p=0}^{[(n+1)/2]} \alpha_p^n\, \rho^{2p}$ for $G_n(\xi)$. The limits in (4.20) are quite easy to calculate. However, we suppose the relations (4.14) for the polynomials $R_k^n$ may be used here to find an iterative algorithm for these limits. For the first one set

$$A_n(a) = \sum_{k=0}^{n} R_k^n(e^{-a}).$$

Using the relation $\frac{d}{da} R_k^n(e^{-a}) = -e^{-a}\, R_k^{n\,\prime}(e^{-a})$ we obtain from (4.14):

$$A_{n+1}(a) = \left[1 + (2n+1)\,e^{-a} - 2(n+1)\,e^{-2a}\right] A_n(a) - (1-e^{-a})^2 A_n'(a). \tag{4.21}$$

Assume that the relation holds

$$A_n^{(j)}(0) \begin{cases} = 0 & \text{for } j = 0, \ldots, n, \\ \neq 0 & \text{for } j = n+1, \end{cases} \tag{4.22}$$

which is true for $n = 1$. For the derivatives of $A_{n+1}$ we obtain from (4.21):

$$A_{n+1}^{(j)}(a) = \sum_{i=0}^{j} \binom{j}{i} \left\{ \frac{d^i}{da^i} \left[1 + (2n+1)\, e^{-a} - 2(n+1)\, e^{-2a}\right] \cdot A_n^{(j-i)}(a) \right.$$
$$\left. - \frac{d^i}{da^i} \left[(1-e^{-a})^2\right] \cdot A_n^{(j+1-i)}(a) \right\}.$$

When $a$ is equal to 0, the expression on the right–hand side vanishes for $j < n$ by the assumption. The terms $1 + (2n+1)\, e^{-a} - 2(n+1)\, e^{-2a}\big|_{a=0}$ and $(1-e^{-a})^2\big|_{a=0}$ are equal to 0; the first derivative of $(1-e^{-a})^2$ also vanishes for $a = 0$ and therefore $A_{n+1}^{(n)}(0) = A_{n+1}^{(n+1)}(0) = 0$. For $j = n+2$ we obtain

$$\begin{aligned} A_{n+1}^{(n+2)}(a)\Big|_{a=0} &= \left[\binom{n+2}{1} \frac{d}{da} \left[1 + (2n+1)\, e^{-a} - 2(n+1)\, e^{-2a}\right] \right. \\ &\quad \left. - \binom{n+2}{2} \frac{d^2}{da^2} \left[(1-e^{-a})^2\right]\right] A_n^{(n+1)}(a)\Big|_{a=0} \\ &= (n+2)^2 A_n^{(n+1)}(0). \end{aligned}$$

This proves the relation (4.22) for $A_{n+1}$. Obviously, (4.22) holds also for $f_n : a \mapsto a^{n+1}$ instead of $A_n$, with $f_n^{(n+1)}(0) = (n+1)!$, and hence the 0th coefficients of $G_n$ satisfy the relation

$$\alpha_0^{n+1} = (n+2) \cdot \alpha_0^n.$$

For $p \geq 1$ set $A_n^p(a) = \sum_{k=1}^n c_{k,p}\, R_k^n(e^{-a})$. Then we have

$$\begin{aligned} A_{n+1}^p(a) &= \left[1 - 2(n+1)\, e^{-2a}\right] A_n^p(a) - (1 - e^{-2a})\, A_n^{p\prime}(a) \\ &\quad + (2n+1)\, e^{-a}\, B_n^p(a) + 2e^{-a}\, B_n^{p\prime}(a), \end{aligned}$$

where $B_n^p(a)$ stays for $\sum_{k=0}^n c_{k+1,p}\, R_k^n(e^{-a})$. Remember that $c_{k,p}$ were the coefficients of the power series of $\left(\frac{1-\cdot}{1+\cdot}\right)^k$, hence, they satisfy the relation

$$\sum_{p=0}^{\infty} c_{k+1,p}\, t^p = \left(\sum_{p=0}^{\infty} c_{k,p}\, t^p\right) \cdot \left(\sum_{p=0}^{\infty} c_{1,p}\, t^p\right),$$

and therefore

$$c_{k+1,p} = \sum_{r=0}^{p} c_{k,r} \cdot c_{1,p-r}.$$

We then can write $B_n^p(a)$ as $\sum_{r=0}^p c_{1,p-r}\, A_n^r(a) + c_{1,p} \cdot 1 \cdot R_0^n(e^{-a})$ and for $A_{n+1}^p$

Table 4.2: Euklidean limits for the Poisson wavelets of order 1 to 9

| | |
|---|---|
| $g^1(\xi) =$ | $\frac{2-\lvert\xi\rvert^2}{2\pi\,(1+\lvert\xi\rvert^2)^{5/2}}$ |
| $g^2(\xi) =$ | $\frac{3\,(2-3\lvert\xi\rvert^2)}{2\pi\,(1+\lvert\xi\rvert^2)^{7/2}}$ |
| $g^3(\xi) =$ | $\frac{3\,(8-24\lvert\xi\rvert^2+3\lvert\xi\rvert^4)}{2\pi\,(1+\lvert\xi\rvert^2)^{9/2}}$ |
| $g^4(\xi) =$ | $\frac{15\,(8-40\lvert\xi\rvert^2+15\lvert\xi\rvert^4)}{2\pi\,(1+\lvert\xi\rvert^2)^{11/2}}$ |
| $g^5(\xi) =$ | $\frac{45\,(16-120\xi^2+90\xi^4-5\xi^6)}{2\pi\,(1+\lvert\xi\rvert^2)^{13/2}}$ |
| $g^6(\xi) =$ | $\frac{315\,(16-168\xi^2+210\xi^4-35\xi^6)}{2\pi\,(1+\lvert\xi\rvert^2)^{15/2}}$ |
| $g^7(\xi) =$ | $\frac{315\,(128-1792\xi^2+3360\xi^4-1120\xi^6+35\xi^8)}{2\pi\,(1+\lvert\xi\rvert^2)^{17/2}}$ |
| $g^8(\xi) =$ | $\frac{2835\,(128-2304\xi^2+6048\xi^4-3360\xi^6+315\xi^8)}{2\pi\,(1+\lvert\xi\rvert^2)^{19/2}}$ |
| $g^9(\xi) =$ | $\frac{14175\,(256-5760\xi^2+20160\xi^4-16800\xi^6+3150\xi^8-63\xi^{10})}{2\pi\,(1+\lvert\xi\rvert^2)^{21/2}}$ |

we obtain the following expression:

$$\begin{aligned} A^p_{n+1}(a) &= \left[1-2(n+1)\,e^{-2a}\right]\,A^p_n(a) - (1-e^{-2a})\,A^{p\,\prime}_n(a) \\ &\quad + (2n+1)\,e^{-a}\left(\sum_{r=0}^{p} c_{1,p-r}\,A^r_n(a) + c_{1,p}\,R^n_0(e^{-a})\right) \\ &\quad + 2e^{-a}\left(\sum_{r=0}^{p} c_{1,p-r}\,A^{r\,\prime}_n(a) - c_{1,p}\,e^{-a}\,R^{n\,\prime}_0(e^{-a})\right). \end{aligned}$$

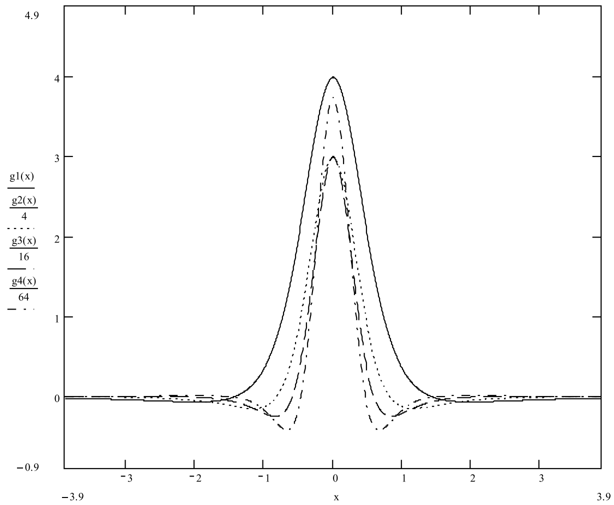
For further investigation the relation

$$\left.\frac{d^j}{da^j}A^p_n(a)\right|_{a=0} \begin{cases} =0 & j=0,\ldots,n-2p, \\ \neq 0, & j=n+1-2p, \end{cases}$$

(for $n \geq 2p$) could be exploited, however, we were not able to obtain any satisfying results.

Formulae for Euklidean limits of some wavelets of higher order are collected in Table 4.2. The limits of the first four wavelets are depicted in Figure 4.5.

Figure 4.5: Euklidean limits for the wavelets of order 1 to 9




Note the interesting behaviour of $g^n$ at infinity. The functions decay polynomially with degree $\iota_n = n + 2 + (n+1)_{\mathrm{mod}2}$. This exponent is the difference between $2\left[(n+1)/2\right]$, the highest exponent by $\rho$ in (4.20) (which is indeed different from 0), and $2(n+3/2)$ from (4.18).

## Integral of the Euklidean limit

For any scale $a$, the wavelet $g_a^n$ is of zero–mean. This is a strightforward consequence of the fact that the wavelets are series of reproducing kernels $Q_l$, which are of zero–mean. We would like to show that the same holds for the Euklidean limit of $g_a^n$. Obviously, since $g^n$ is a function over $\mathbb{R}^2$, we need a measure $\nu$ that reflects the properties of the sphere, i.e., such that

$$\int_{\mathbb{R}^2} f(\xi)\, d\nu(\xi) = \int_{\Omega} f\left(\Phi^{-1}(\xi)\right) d\omega(\Phi^{-1}(\xi)).$$

Since $\Phi^{-1}(\rho,\phi) = (2\arctan\frac{\rho}{2},\phi)$, $\xi = (\rho,\phi) \in \mathbb{R}^2$, $x = (\theta,\phi) \in \Omega$, we obtain by a direct calculation

$$d\nu(\xi) = \frac{16\rho}{(4+\rho^2)^2}\, d\rho\, d\phi.$$

Now, for any $a$, $g_a^n$ is a continuous function on a compact domain, hence, it is bounded by a constant. Similarly, the Euklidean limit is a bounded function (since it is continuous and vanishes in infinity.) Therefore, one can find a common bound $\mathfrak{c}$ for $a^2 g_a^n\left(\Phi^{-1}(a\,\cdot)\right)$, $a \in (0,1]$, and the Euklidean limit $g^n$. This is an integrable majorant with respect to the measure $\nu$. According to the

Lebesgue domiated convergence theorem, the Euklidean limit $g^n$ has the same mean as the wavelets $g_a^n$, that is, zero.

## 4.5 Localization of $g_a^n$ and its surface gradient

### Localization of the wavelet

We have seen in the previous section that the Euklidean limit of the wavelets behaves like

$$g^n(\xi) \sim \frac{1}{|\xi|^{\iota_n}}, \tag{4.23}$$

with $\iota_n = n+2+(n+1)_{\mathrm{mod}\,2}$, for arguments with big radii. Now, we would like to investigate the behaviour of the wavelets on a scale $a$: does a similar relation

$$g_a^n\left((\theta,\phi)\right) = \mathcal{O}\left((\theta/a)^{-\kappa_n}\right), \quad \theta \to \pi, \tag{4.24}$$

hold? The answer is positive, but the exponents $\kappa_n$ differ from those for the Euklidean limit if $n$ is an even number. The formula is

$$\kappa_n = n+2. \tag{4.25}$$

This means that for even $n$, the decrease of the Euklidean limits is of one power stronger that the decrease of the wavelets themself (but don't forget that the wavelets and their Euklidean limits live on different domains.)
In order to prove the relation (4.24) with exponent given by (4.25), we need a technical lemma that describes behaviour of the polynomial part $\sum_{k=0}^n R_k^n(\lambda)\,\cos^k\theta$ in the representation (4.10) of a wavelet.

**Lemma 4** *Let $Q_n$, $n \in \mathbb{N}$, be a sequence of polynomials in two variables satisfying the recusion*

$$Q_{n+1}(\lambda,y) = a_n(\lambda,y)\cdot Q_n(\lambda,y) + b(\lambda,y)\cdot \frac{\partial}{\partial\lambda}Q_n(\lambda,y) \tag{4.26}$$

*with*

$$a_n(\lambda,y) = 1-2(n+1)\lambda^2+(2n+1)\lambda y \quad \text{and} \quad b(\lambda,y) = (1+\lambda^2-2\lambda y)\lambda,$$

*and such that*

$$Q_1(1,1) = 0, \quad \text{and} \quad \frac{\partial}{\partial\lambda}\,Q_1(\lambda,1)\bigg|_{\lambda=1} \neq 0. \tag{4.27}$$

*Then the polynomial $Q_n(1,\cdot)$, $n \geq 2$, has an $[(n+1)/2]$–fold root in 1.*

*Proof.* First, we shall prove by induction that

$$\begin{aligned} &\frac{\partial^k}{\partial\lambda^k}\,Q_n(\lambda,1)\bigg|_{\lambda=1} = 0 \quad \text{for } k=0,1,\dots,n-1,\\ \text{and} \quad &\frac{\partial^n}{\partial\lambda^n}\,Q_n(\lambda,1)\bigg|_{\lambda=1} \neq 0. \end{aligned} \tag{4.28}$$

Let $q_n = Q_n(\cdot, 1)$; for $n = 1$ we have

$$q_1(1) = 0 \quad \text{and} \quad q_1'(1) \neq 0$$

by assumption. Suppose, for some $n$,

$$q_n^{(k)}(1) = 0 \quad \text{for } k = 0, 1, \ldots, n-1, \quad q_n^{(n)}(1) \neq 0. \tag{4.29}$$

We rewrite the relation (4.26) in the form

$$q_{n+1}(\lambda) = a_n(\lambda) \cdot q_n(\lambda) + b(\lambda) \cdot q_n'(\lambda)$$

with

$$a_n(\lambda) = 1 + (2n+1)\lambda - 2(n+1)\lambda^2 \quad \text{and} \quad b(\lambda) = (1-\lambda)^2 \lambda.$$

Then, the $k^{\text{th}}$ derivative of $q_{n+1}$ is given by

$$q_{n+1}^{(k)}(\lambda) = \sum_{j=0}^{k} \binom{k}{j} \left[ a_n^{(j)}(\lambda) \cdot q_n^{(k-j)}(\lambda) + b^{(j)}(\lambda) \cdot q_n^{(k-j+1)}(\lambda) \right],$$

and since only the first and the second derivative of $a_n$ and only the second and the third derivative of $b$ do not vanish in $\lambda = 1$, we obtain

$$\begin{aligned} &q_{n+1}(1) = 0, \quad q_{n+1}'(1) = a_n'(1) \cdot q_n(1) \quad \text{and} \\ &q_{n+1}^{(k)}(1) = \binom{k}{1} a_n'(1) \cdot q_n^{(k-1)}(1) + \binom{k}{2} a_n''(1) \cdot q_n^{(k-2)}(1) \\ &\qquad + \binom{k}{2} b''(1) \cdot q_n^{(k-1)}(1) + \binom{k}{3} b'''(1) \cdot q_n^{(k-2)}(1) \quad \text{for } k \geq 2. \end{aligned}$$

Consequently, $q_{n+1}^{(k)}(1) = 0$ for $k \leq n$ and

$$\begin{aligned} q_{n+1}^{(n+1)}(1) &= (n+1) \cdot \left[ a_n'(1) + \frac{n}{2} \cdot b''(1) \right] \cdot q_n^{(n)}(1) \\ &= -(n+1)(n+3)\, q_n^{(n)}(1) \neq 0. \end{aligned}$$

Now, using the relation (4.29) we are able to prove that

$$\begin{aligned} \frac{\partial^k}{\partial y^k} Q_n(1, y) \bigg|_{y=1} &= 0 \quad \text{for } k = 0, 1, \ldots, \left[ \frac{n+1}{2} \right] - 1, \\ \frac{\partial^k}{\partial y^k} Q_n(1, y) \bigg|_{y=1} &\neq 0 \quad \text{for } k = \left[ \frac{n+1}{2} \right] \end{aligned}$$

for $n \geq 2$. The formula (4.26) yields

$$\begin{aligned} &\frac{\partial^k}{\partial y^k} Q_n(1, y) \bigg|_{y=1} = \frac{\partial^k}{\partial y^k} Q_n(\lambda, y) \bigg|_{\substack{\lambda=1 \\ y=1}} \\ &= \sum_{j=0}^{k} \binom{k}{j} \left[ \frac{\partial^j}{\partial y^j} a_{n-1}(\lambda, y) \cdot \frac{\partial^{k-j}}{\partial y^{k-j}} Q_{n-1}(\lambda, y) + \frac{\partial^j}{\partial y^j} b(\lambda, y) \cdot \frac{\partial^{k-j}}{\partial y^{k-j}} \frac{\partial}{\partial \lambda} Q_{n-1}(\lambda, y) \right]_{\substack{\lambda=1 \\ y=1}}. \end{aligned}$$

(Since $Q_{n-1}$ is a $\mathcal{C}^\infty$–function, we can exchange the differentiation and limit, as well as the order of the differentiation and the order of limit computation.) The polynomials $a_{n-1}$ and $b$ are linear in $y$, further, $a_{n-1}(1,1) = b(1,1) = 0$, and therefore the terms with $j \neq 1$ vanish. We obtain immediately $Q_n(1,1) = 0$. Further, for $k \geq 1$,

$$\begin{aligned} \frac{\partial^k}{\partial y^k} Q_n(1,y)\bigg|_{y=1} &= k \left[\frac{\partial}{\partial y} a_{n-1}(\lambda,y) + \frac{\partial}{\partial y} b(\lambda,y) \frac{\partial}{\partial \lambda}\right] \frac{\partial^{k-1}}{\partial y^{k-1}} Q_{n-1}(\lambda,y)\bigg|_{\substack{\lambda=1\\y=1}} \\ &= k\lambda \left(2n-1-2\lambda\frac{\partial}{\partial\lambda}\right) \frac{\partial^{k-1}}{\partial y^{k-1}} Q_{n-1}(\lambda,y)\bigg|_{\substack{\lambda=1\\y=1}} . \end{aligned} \tag{4.30}$$

The last equation means, we were able to reduce the order of differentiation in $y$ and the index of the polynomial; however, one more diferentiation in $\lambda$ is needed. A $k$–fold application of this procedure yields:

$$\frac{\partial^k}{\partial y^k} Q_n(1,y)\bigg|_{y=1} = \sum_{j=0}^{k} c_j(\lambda) \cdot \frac{\partial^j}{\partial \lambda^j} Q_{n-k}(\lambda,1)\Bigg|_{\lambda=1}, \tag{4.31}$$

where $c_j$ are some polynomials. If $k \leq [(n+1)/2]-1$, then $2k \leq n+1-2 = n-1$, and further, $k \leq (n-k)-1$. Therefore, all the derivatives on the right–hand–side of (4.31) vanish, and consequently $\frac{\partial^k}{\partial y^k} Q_n(1,y)\big|_{y=1} = 0$. For $k = [(n+1)/2]$ one has $k \geq n-k$ and hence $\frac{\partial^k}{\partial \lambda^k} Q_{n-k}(\lambda,1)\big|_{\lambda=1} \neq 0$. The polynomial $c_k(\lambda)$ is equal to $(-2)^k\lambda^{2k}k!$ (compare (4.30)), i.e., different from zero in $\lambda = 1$. This yields $\frac{\partial^{[(n+1)/2]}}{\partial y^{[(n+1)/2]}} Q_n(1,y)\big|_{y=1} \neq 0$. □

**Lemma 5** *Let $f_n$ be a family of functions given by*

$$\begin{aligned} f_1(\lambda,\theta) &= \frac{\lambda\, Q_1(\lambda,\cos\theta)}{(1+\lambda^2-2\lambda\cos\theta)^{3/2}}, \\ f_{n+1}(\lambda,\theta) &= \lambda\frac{\partial}{\partial\lambda} f_n(\lambda,\theta), \end{aligned} \tag{4.32}$$

*where $Q_1$ is a polynomial satisfying* (4.27)*. Then, for any $k \geq 2[n/2]+1$ and any $\lambda_0 \in (0,1)$ there exists a constant $\mathfrak{c}$ such that*

$$|f_n(\lambda,\theta)| \leq \frac{\mathfrak{c}}{\theta^k}, \quad \theta \in (0,\pi], \tag{4.33}$$

*uniformly in $\lambda$ for $\lambda \in [\lambda_0, 1)$. For $n \geq 2$, the number $2[n/2]+1$ is the smallest possible exponent $k$. If $Q_1(1,y)$ has a simple root in* 1*, then* 1 *is the smallest possible exponent $k$ on the right–hand–side of* (4.33) *for $n = 1$.*

*Proof.* For any $n \in \mathbb{N}$, the function $f_n$ is given by

$$f_n(\lambda,\theta) = \frac{\lambda\, Q_n(\lambda,\cos\theta)}{(1+\lambda^2-2\lambda\cos\theta)^{n+1/2}}, \tag{4.34}$$

where $Q_n$ is a polynomial obtained recursivly via (4.26) (for the derivation of this formula compare Sec. 4.2, p. 32.) Consider the function $\tilde{f} : \theta \mapsto \theta^k f(\lambda, \theta)$ and define $F : [\lambda_0, 1] \times [0, \pi]$ by

$$F(\lambda, \theta) = \begin{cases} \theta^k f_n(\lambda, \theta), & \lambda < 1, \\ \frac{\theta^k}{[2(1-\cos\theta)]^{n+1/2}} Q_n(1, \cos\theta), & \lambda = 1, \theta > 0, \\ 0, & \lambda = 1, \theta = 0. \end{cases}$$

Since $\lim_{\theta \to 0^+} \frac{2(1-\cos\theta)}{\theta^2} = 1$, one has

$$\lim_{\theta \to 0} F(1, \theta) = \lim_{\theta \to 0} \frac{\theta^k}{\theta^{2(n+1/2)}} \cdot \theta^{2[(n+1)/2]} \cdot \frac{Q_n(1, \cos\theta)}{(1-\cos\theta)^{[(n+1)/2]}}.$$

The sum of powers of $\theta$ is equal to $k - (2[n/2] + 1) \geq 0$; the limit of the last fraction exists according to the previous lemma. Therefore, the function $F$ is continuous. It is a continuous extension of $\tilde{f}$ to the compact set $[\lambda_0, 1] \times [0, \pi]$. Consequently, the function $\tilde{f}$ is bounded; this yields the desired inequality (4.33). For the minimality of $k$ note that $[(n+1)/2]$ (multiplicity of the root of $Q_n(1, y)$ in $y = 1$) is the biggest possible exponent in the last fraction that ensures that the limit of the fraction exists; further, $\theta^k / \theta^{2(n+1/2)} \cdot \theta^{2[(n+1)/2]}$ would be divergent for $\theta \to 0$ if $k < 2[n/2] + 1$. □

Note that the critical point is around 0. For arguments $\theta$ far from 0 the inequality (4.39) is valid for any $k$.

Functions that satisfy the conditions of the lemma are, e.g., those describing field caused by a multipole inside the Earth.

**Lemma 6** *Let*

$$\Psi_\lambda^n : (\theta, \phi) \mapsto \frac{1}{4\pi} \sum_{l=0}^{\infty} P_l(\cos\theta) l^n \lambda^l, \; n \in \mathbb{N}_0,$$

*be the field on the sphere caused by the multipole (monopole for $n = 0$) $\mu = (\lambda\partial_\lambda)^n \delta_{\lambda\hat{e}}$. For any $k \geq 2[n/2] + 1$ and any $\lambda \in (0, 1)$ there exists a constant $\mathfrak{c}$ such that*

$$|\Psi_\lambda^n(\theta, \phi)| \leq \frac{\mathfrak{c}}{\theta^k}, \quad \theta \in (0, \pi], \tag{4.35}$$

*uniformly in $\lambda$ for $\lambda \in [\lambda_0, 1)$. $2[n/2] + 1$ is the smallest possible exponent on the right–hand–side of this inequality.*

*Proof.* The first multipole is given by

$$\Psi_\lambda^1(\theta, \phi) = \frac{\lambda}{(1 + \lambda^2 - 2\lambda\cos\theta)^{3/2}} \cdot Q_1(\lambda, \cos\theta),$$

with $Q_1(\lambda, y) = (y - \lambda)/4\pi$. This polynomial satisfies the conditions (4.27) and its restriction to $\lambda = 1$ has a simple root in $y = 1$. Further

$$\Psi_\lambda^{n+1} = \lambda \frac{\partial}{\partial \lambda} \Psi_\lambda^n.$$

Thus, the previous lemma applies and (4.35) holds for $n \geq 1$. For $n = 0$ the estimation may be proven in the same way as in the last lemma using the direct representation of the monopole:

$$\Psi_\lambda^0(\theta, \phi) = \frac{1}{4\pi} \cdot \frac{1}{\sqrt{1 + \lambda^2 - 2\lambda \cos\theta}}.$$

□

Now we may come to the localization of Poisson wavelets.

**Theorem 2** *Let $g_a^n$ be a Poisson wavelet family of order $n$. For any positive $a_0$ there exists a constant $\mathfrak{c}$ such that*

$$|a^2 g_a^n(a\theta)| \leq \frac{\mathfrak{c}}{\theta^{n+2}}, \quad \theta \in \left(0, \frac{\pi}{a}\right], \tag{4.36}$$

*uniformly in $a$ for $a \leq a_0$. $n+2$ is the biggest possible exponent in this inequality.*

*Proof.* The function

$$f_n : (a, \theta) \mapsto g_a^n(\theta)/a^n$$

can be expressed as a sum of fields caused by multipoles, $f_n(a, \theta) = 2\Psi_{e^{-a}}^{n+1}(\theta, \phi) + \Psi_{e^{-a}}^n(\theta, \phi)$, see Sec. 4.1. Since $n+2 \geq 2\,[(n+1)/2]+1 \geq 2[n/2]+1$ and according to the last lemma, the relation

$$|f_n(a, \theta)| = \left|\frac{a^2 g_a^n(\theta)}{a^{n+2}}\right| \leq \frac{\mathfrak{c}}{\theta^{n+2}}, \quad \theta \in [0, \pi],$$

holds uniformly in $a$ for $a \in (0, a_0]$, $a_0 := -\log \lambda_0$. Upon replacing $\theta$ by $a\theta$ and multiplying both sides by $a^{n+2}$, we obtain the desired inequality. For the second statement note that $f_n(0, \theta)$ is a non–vanishing function of $\theta$ and therefore $\left(\frac{\theta}{a}\right)^\epsilon f_n(a, \theta)$ diverges for $a \to 0$ for any positive exponent $\epsilon$; thus, $(a, \theta) \mapsto (\theta/a)^{n+2+\epsilon} a^2 g_a^n(a\theta)$ is not bounded. □

*Remark.* This theorem may be proven directly with use of Lemma 5. One chooses $Q_1(\lambda, y) = 1 - \lambda^2$, then, $f_{n+1}(e^{-a}, \theta) = g_a^n(\theta)/a^n$.

**Corollary 1** *The functions $(a, \theta) \mapsto a^2 g_a^n(a\theta)$ are uniformly bounded in $\theta$ for $a \leq a_0$.*

*Proof.* Set $k = 0$ in (4.36). □

This property has already been used in Section 4.4.

Similarly, we can show that for big values of $a\theta$ the function $|a^2 g_a^n(a\theta)|$ is bounded from below by $\mathfrak{c}/\theta^{n+2}$. More exactly, we have the following

**Proposition 1** *Let $g_a^n$ be a Poisson wavelet of order $n \le 100$. Then there exist constants $a_0$, $\theta_0$ and $\mathfrak{c}$ such that*

$$|a^2 g_a^n(a\theta)| \ge \frac{\mathfrak{c}}{\theta^{n+2}}, \quad \theta \in \left[\frac{\theta_0}{a}, \frac{\pi}{a}\right],$$

*uniformly in $a$ for $a \le a_0$.*

*Proof.* Consider the function

$$f_n : (0,1] \times [0,\pi] \to \mathbb{R}, \qquad (a,\theta) \mapsto (\theta/a)^{n+1} a^2 g_a^{n-1}(\theta)$$

for $n \ge 2$. Its continuous extension to the set $[0,1] \times [0,\pi]$ is given by

$$F_n(a,\theta) = \begin{cases} f_n(a,\theta), & a > 0, \\ \frac{\theta^{n+1}}{[2(1-\cos\theta)]^{n+1/2}} Q_n(1,\cos\theta), & a = 0,\, \theta > 0, \\ 0, & a = 0,\, \theta = 0. \end{cases}$$

with polynomials $Q_n$ as decribed in the previous lemma, $\lambda = e^{-a}$. Since $F_n(\cdot,\pi)$ is a polynomial, it has a dicrete set of zeros. We have numerically tested that $F_n(0,\pi) \ne 0$ for $n \le 101$. Therefore, for any index $n \le 101$ there exists $a_0 = a_0(n)$ (less than the smallest positive zero of $F_n(\cdot,\pi)$) such that $F_n(\cdot,\pi)$ has no roots on $[0,a_0]$. Since $F_n$ is continuous, the function

$$h_n : [0,a_0] \to [\pi/2,\pi], \qquad a \mapsto \sup\{\vartheta \in [0,\pi] : F_n(a,\vartheta) = 0 \vee \vartheta = \pi/2\}$$

is well–defined and continuous; its supremum exists and is less than $\pi$. Set $\theta_0 = \sup_{a\in[0,a_0]} h(a) + \epsilon$, with $\epsilon > 0$ and such that $\theta_0 < \pi$. Then, the function $F_n$ restricted to the compact set $[0,a_0] \times [\theta_0,\pi]$ does not vanish, and hence

$$|F_n(a,t)| \ge \mathfrak{c} \quad \text{uniformly in } [0,a_0] \times [\theta_0,\pi]$$

for some constant $\mathfrak{c} > 0$. This yields the desired inequality. □

Note that we investigate the behaviour of $g_a^n(a\theta)$, whereas in the Euklidean limit one has the expression $g_a^n\left(\Phi^{-1}(a\rho)\right)$ (where $\Phi^{-1}$ is understood as a function of radius in spherical coordinates.) The inequality

$$a^2 g_a^n\left(\Phi^{-1}(a\rho)\right) \le \frac{\mathfrak{c}}{\rho^{n+2}}$$

corresponding to (4.36) does not hold, we mererely have

$$a^2 g_a^n\left(\Phi^{-1}(a\rho)\right) = a^2 g_a^n\left(2\arctan\frac{a\rho}{2}\right) \le \frac{\mathfrak{c}\, a^{n+2}}{[2\arctan(a\rho/2)]^{n+2}},$$

and for $\rho$ tending to infinity, the last fraction tends to $\mathfrak{c}\,(a/\pi)^{n+2}$, i.e., does not vanish.

## Localization of the colatitudinal derivative of the wavelet

Analogous statements can be made for the colatitudinal derivative of the wavelet $g_a^n$. Since the longitudinal one is equal to zero, we immediately have

$$|\nabla_* g_a^n(\theta,\phi)| \le \left|\frac{\partial}{\partial\theta} g_a^n(\theta,\phi)\right|$$

with

$$\nabla_* g_a^n(\theta,\psi) = \left(\frac{\partial}{\partial\theta} g_a^n(\theta,\psi), \frac{\partial}{\partial\phi} g_a^n(\theta,\phi)\right).$$

**Lemma 7** *Let $f_n$ be a family of functions given by*

$$f_1(\lambda,\theta) = \frac{\lambda \sin\theta\, Q_1(\lambda,\cos\theta)}{(1+\lambda^2-2\lambda\cos\theta)^{5/2}}, \tag{4.37}$$
$$f_{n+1}(\lambda,\theta) = \lambda\frac{\partial}{\partial\lambda} f_n(\lambda,\theta),$$

*where $Q_1$ is a polynomial satisfying (4.27). Then, for any $k \ge 2[n/2]+2$ and any $\lambda_0 \in (0,1)$ there exists a constant $\mathfrak{c}$ such that*

$$|f_n(\lambda,\theta)| \le \frac{\mathfrak{c}}{\theta^k}, \quad \theta \in (0,\pi], \tag{4.38}$$

*uniformly in $\lambda$ for $\lambda \in [\lambda_0, 1)$. For $n \ge 2$, the number $2[n/2]+2$ is the smallest possible exponent $k$. If $Q_1(1,y)$ has a simple root in 1, then 1 is the smallest possible exponent $k$ on the right–hand–side of (4.38) for $n=1$.*

*Proof.* For any $n \in \mathbb{N}$, the function $f_n$ is given by

$$f_n(a,\theta) = \frac{\lambda \sin\theta\, Q_n(\lambda,\cos\theta)}{(1+\lambda^2-2\lambda\cos\theta)^{n+3/2}}, \tag{4.39}$$

where $Q_n$ is a polynomial obtained recursivly via (4.26). Consider the function $\tilde{f} : \theta \mapsto \theta^k f(\lambda,\theta)$ and define $F : [\lambda_0, 1] \times [0,\pi]$ by

$$F(\lambda,\theta) = \begin{cases} \theta^k f_n(\lambda,\theta), & \lambda < 1, \\ \frac{\theta^k \sin\theta}{[2(1-\cos\theta)]^{n+3/2}}\, Q_n(1,\cos\theta), & \lambda = 1,\ \theta > 0, \\ 0, & \lambda = 1,\ \theta = 0. \end{cases}$$

Since $\lim_{\theta\to 0^+} \frac{2(1-\cos\theta)}{\theta^2} = \lim_{\theta\to 0^+} \frac{[2(1-\cos\theta)]^{1/2}}{\sin\theta} = 1$, one has

$$\lim_{\theta\to 0} F(1,\theta) = \lim_{\theta\to 0} \frac{\theta^k}{\theta^{2(n+1)}} \cdot \theta^{2[(n+1)/2]} \cdot \frac{Q_n(1,\cos\theta)}{(1-\cos\theta)^{[(n+1)/2]}}.$$

The sum of powers of $\theta$ is equal to $k - (2[n/2]+2) \ge 0$; the limit of the last fraction exists according to the Lemma 4. Thus, the function $F$ is a continuous extension of $\tilde{f}$ to the compact set $[\lambda_0, 1] \times [0, \pi]$, and further, the function $\tilde{f}$ is bounded. This yields the desired inequality (4.38). For the minimality of $k$ note that $[(n+1)/2]$ (multiplicity of the root of $Q_n(1, y)$ in $y = 1$) is the biggest possible exponent in the last fraction that ensures that the limit of the fraction exists; further, $\theta^k/\theta^{2(n+1)} \cdot \theta^{2[(n+1)/2]}$ would be divergent for $\theta \to 0$ if $k < 2[n/2] + 2$. □

Again, this lemma may be applied to (colatitudinal dervative of) the field caused by a multipole.

**Lemma 8** *Let*

$$\Psi_\lambda^n : (\theta, \phi) \mapsto \frac{1}{4\pi} \sum_{l=0}^{\infty} P_l(\cos\theta) l^n \lambda^l, \; n \in \mathbb{N}_0,$$

*be the field on the sphere caused by the multipole (monopole for $n = 0$) $\mu = (\lambda\partial_\lambda)^n \delta_{\lambda\hat{e}}$. For any $k \geq 2[n/2] + 2$ and any $\lambda \in (0, 1)$ there exists a constant $\mathfrak{c}$ such that*

$$\left| \frac{\partial}{\partial\theta} \Psi_\lambda^n(\theta, \phi) \right| \leq \frac{\mathfrak{c}}{\theta^k}, \quad \theta \in (0, \pi], \tag{4.40}$$

*uniformly in $\lambda$ for $\lambda \in [\lambda_0, 1)$. $2[n/2] + 2$ is the smallest possible exponent on the right–hand–side of this inequality.*

*Proof.* For the first multipole we have

$$\frac{\partial}{\partial\theta} \Psi_\lambda^1(\theta, \phi) = \frac{\lambda \sin\theta \, Q_1(\lambda, \cos\theta)}{(1 + \lambda^2 - 2\lambda\cos\theta)^{5/2}},$$

with $Q_1(\lambda, y) = \lambda(-1 + 2\lambda^2 - \lambda\cos\theta)/4\pi$. This polynomial satisfies the conditions (4.27) and its restriction to $\lambda = 1$ has a simple root in $y = 1$. Further

$$\frac{\partial}{\partial\theta} \Psi_\lambda^{n+1}(\theta, \phi) = \lambda \frac{\partial}{\partial\lambda} \left( \frac{\partial}{\partial\theta} \Psi_\lambda^n(\theta, \phi) \right).$$

Thus, the previous lemma applies and (4.40) holds for $n \geq 1$. For $n = 0$ the estimation may be proven in the same way as in the previous lemma, with use of the representation:

$$\frac{\partial}{\partial\theta} \Psi_\lambda^0(\theta, \phi) = \frac{1}{4\pi} \cdot \frac{-\lambda\sin\theta}{(1 + \lambda^2 - 2\lambda\cos\theta)^{3/2}}.$$

□

And now we may come to the most important result in this subsection, the localization of Poisson wavelets.

**Theorem 3** *Let $g_a^n$ be a Poisson wavelet family of order $n$. For any positive $a_0$ there exists a constant $\mathfrak{c}$ such that*

$$\left| a^3 \frac{\partial}{\partial\theta} g_a^n(\theta) \right|_{\theta = a\theta} \leq \frac{\mathfrak{c}}{\theta^{n+3}}, \quad \theta \in \left(0, \frac{\pi}{a}\right], \tag{4.41}$$

*uniformly in $a$ for $a \leq a_0$. $n+3$ is the biggest possible exponent in this inequality.*

*Proof.* The function

$$f_n : (a, \theta) \mapsto \frac{1}{a^n} \frac{\partial}{\partial \theta} g_a^n(\theta)$$

can be written as

$$f_n(a, \theta) = 2 \frac{\partial}{\partial \theta} \Psi_{e^{-a}}^{n+1}(\theta, \phi) + \frac{\partial}{\partial \theta} \Psi_{e^{-a}}^{n}(\theta, \phi)$$

Since $n + 3 \geq 2\,[(n+1)/2] + 2 \geq 2[n/2] + 2$ and according to the last lemma, the relation

$$|f_n(a, \theta)| = \frac{1}{a^n}\, |g_a^n{}'(\theta)| \leq \frac{\mathfrak{c}}{\theta^{n+3}}, \quad \theta \in [0, \pi],$$

holds uniformly in $a$ for $a \in (0, a_0]$, $a_0 := -\log \lambda_0$. Upon replacing $\theta$ by $a\theta$ and multiplying both sides by $a^{n+3}$, we obtain the desired inequality. For the second statement note that $f_n(0, \theta)$ is a non–vanishing function of $\theta$ and therefore $\left(\frac{\theta}{a}\right)^{\epsilon} f_n(a, \theta)$ diverges for $a \to 0$ for any positive exponent $\epsilon$; thus, $(a, \theta) \mapsto (\theta/a)^{n+3+\epsilon} a^2 g_a^n(a\theta)$ is not bounded. □

**Corollary 2** *The functions*

$$(a, \theta) \mapsto a^3 \frac{\partial}{\partial \theta} g_a^n(\theta)\Big|_{\theta = a\theta}$$

*are uniformly bounded in $\theta$ for $a \leq a_0$.*

*Proof.* Set $k = 0$ in (4.41).

## 4.6 Convolution properties and reproducing kernels

Since $Q_l$'s are reproducing kernels for the spaces $\Sigma_l$, $l \in \mathbb{N}_0$, we obtain by formula (4.1) the following expression for the convolution of two wavelets:

$$g_a^n * g_b^m = \frac{a^n b^m}{(a+b)^{n+m}}\, g_{a+b}^{n+m}.$$

In particular, the scalar product is given explicitly by

$$\left\langle g_{x,a}^n, g_{y,b}^m \right\rangle = \frac{a^n b^m}{(a+b)^{n+m}}\, g_{a+b}^{n+m}(x \cdot y), \quad x, y \in \Omega.$$

These formulae become simplier if we consider only wavelets of the same order. In this case, $\{g_a^n/\sqrt{c_{\gamma_n}}\}_{a \in \mathbb{R}_+}$ build an admissible analysis–reconstruction wavelet family and the reproducing kernel can be expressed as

$$\Pi_{g^n}(x, a; y, b) = \frac{1}{c_{\gamma_n}} \frac{(ab)^n}{(a+b)^{2n}}\, g_{a+b}^{2n}(x \cdot y).$$

For the $\mathcal{L}^2$–norm we have the formula

$$\|g_a^n\|^2 = 4^{-n} g_{2a}^{2n}(1).$$

# Chapter 5

# Poisson wavelet frames

This chapter contains the most important results of our research. We construct semicontinuous frames of Poisoon wavelets and find some frame bounds of them. Further, we prove that sets of Poisson wavelets constructed by Holschneider *et al.* in [24] and verified numerically to be frames, are indeed some. However, in this case, we leave the computation of frame bounds. It seems to be very complicated and far from optimal if one wants to use analytical methods. As can be seen from the articles of Holschneider *et al.* and Chambodut *et al.* [5], numerical methods are more suited for calculation of the bounds and optimal distribution of sampling points.

## 5.1 Semicontinuous frames

As we have already noticed, the translated and dilated wavelets build a continuous frame. However, it would be nice to find a smaller set of coefficients that contains the whole information about the analysed function. In this section we show that a discretization of the wavelet coefficients with respect to the scale is possible, that means that a semicontinuous frame exists. The next step will be the discretization with respect to the spherical variable.

**Theorem 4** *Let* $\{g_a = \sum_{l=0}^{\infty} \gamma(al) Q_l\}$ *be an admissible family of zonal wavelets and* $\mathcal{A} = \{a_j : j \in \mathbb{N}_0\}$ *— a countable set of scales. The set*

$$\{g_{x,a_j} : x \in \Omega,\, a_j \in \mathcal{A}\} \tag{5.1}$$

*is a semicontinuous frame with weights* $\nu_j$ *and bounds* $A$, $B$ *if and only if*

$$A \le \sum_{j=0}^{\infty} |\gamma(a_j l)|^2 \nu_j \le B \tag{5.2}$$

*holds independently of* $l$.

*Proof.* Suppose, $F$ is a semicontinuous frame with bounds $A$ and $B$. Then the following holds

$$A\|s\|^2 \leq \sum_{j=0}^{\infty} \int_{\Omega} |\mathcal{W}_g s(x,a_j)|^2 \, d\omega(x) \cdot \nu_j \leq B\|s\|^2 \tag{5.3}$$

for any signal $s \in \mathcal{L}^2(\Omega)$. For the wavelet transform of $s$ we have

$$\mathcal{W}_g s(x,a) = \sum_{l=0}^{\infty} \overline{\gamma(al)}\, s_l(x),$$

where $s_l$ is the projection of $s$ onto $\Sigma_l$, see formulae on page 24. Therefore, by the Parseval identity (2.5) we obtain

$$\int_{\Omega} |\mathcal{W}_g s(x,a_j)|^2 \, d\omega(x) = \sum_{l=0}^{\infty} |\gamma(a_j l)|^2 \sum_{m=-l}^{l} |\widehat{s}(l,m)|^2$$

and since all the sums converge absolutely, we may exchange the order of summation with respect to $l$ and with respect to $j$. Consequently, we get

$$A \sum_{\substack{l\in\mathbb{N}_0\\ -l\leq m\leq l}} |\widehat{s}(l,m)|^2 \leq \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} |\gamma(a_j l)|^2 \nu_j \sum_{m=-l}^{l} |\widehat{s}(l,m)|^2 \leq B \sum_{\substack{l\in\mathbb{N}_0\\ -l\leq m\leq l}} |\widehat{s}(l,m)|^2. \tag{5.4}$$

Now, for any $l \in \mathbb{N}_0$ we set $s = Y_l^0$, then we obtain the inequality (5.2).
On the other hand, suppose, (5.2) holds, then by the Parseval inequality and Funk–Hecke formula we obtain from (5.4) the inequality (5.3), and hence, the set (5.1) is a semiframe. □

**Corollary 3** *Let $\{g_a^n\}$ be a Poisson wavelet family of order $n$, $\mathcal{A} = \{a_j\}_{j\in\mathbb{N}_0}$ a decreasing sequence of scales, and $\nu_j = \log(a_j/a_{j+1})$ the corresponding weights. Further, let $\nu_j$, $j \in \mathbb{N}_0$, satisfy*

$$\mathfrak{c}_1 \leq \nu_j \leq \mathfrak{c}_2$$

*for some constants $\mathfrak{c}_1$, $\mathfrak{c}_2 > 0$. Then, $\{g_{x,a_j} : x \in \Omega,\ a_j \in \mathcal{A}\}$ is a semicontinuous frame for $\mathcal{L}^2(\Omega)$.*

*Proof.* The function $\gamma_n : t \mapsto t^n e^{-t}$, as well as $\gamma_n^2$ has its maximum in $n$. Fix a number $l$. In the case $\log(a_0) < \log(n) - \mathfrak{c}_2$ (Fig. 5.1), there exists an index $k$ such that $a_0 \leq a_k l \leq n$ and the sum $\sum_{j=0}^{\infty} \gamma_n^2(a_j l)\nu_j$ is bigger than the integral $\int_0^{a_k l} \gamma_n^2(\alpha)\, d\alpha/\alpha$, consequently, it is bigger than $\int_0^{a_0} \gamma_n^2(\alpha)\, d\alpha/\alpha$. Otherwise (Fig. 5.2), one can find an index $k$ such that $\log(a_k l) \leq \log(n) - \mathfrak{c}_2 \leq \log(a_{k-1} l)$ and

$$\sum_{j=0}^{\infty} \gamma_n^2(a_j l)\nu_j \geq \sum_{j=a_{k-1}}^{\infty} \gamma_n^2(a_j l)\nu_j \geq \int_0^{a_{k-1}l} \gamma_n^2(\alpha)\frac{d\alpha}{\alpha} \geq \int_0^{n/\mathfrak{c}_2} \gamma_n^2(\alpha)\frac{d\alpha}{\alpha}.$$

Figure 5.1: Lower frame–bound, $\log(a_0) < \log(n) - \mathfrak{c}_2$

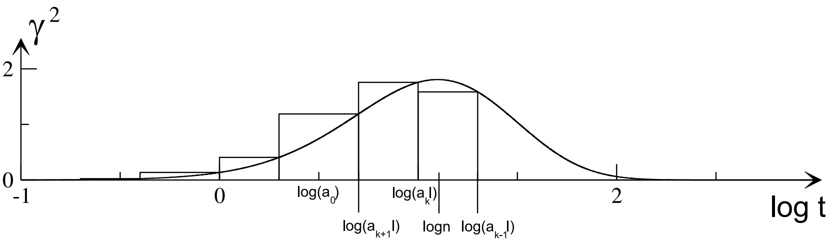
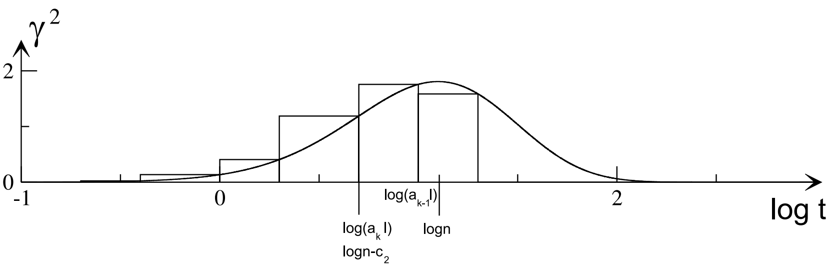



Figure 5.2: Lower frame–bound, $\log(a_0) \geq \log(n) - \mathfrak{c}_2$




Hence, the first inequality in (2.2) is satisfied with

$$A = \min\left\{\int_0^{a_0} \gamma_n^2(\alpha)\frac{d\alpha}{\alpha}, \int_0^{n/\mathfrak{c}_2} \gamma_n^2(\alpha)\frac{d\alpha}{\alpha}\right\}.$$

For the upper bound, let $k$ be again the smallest index such that $\log(a_k l) \leq \log(n) - \mathfrak{c}_2$ (Fig. 5.3.) Then

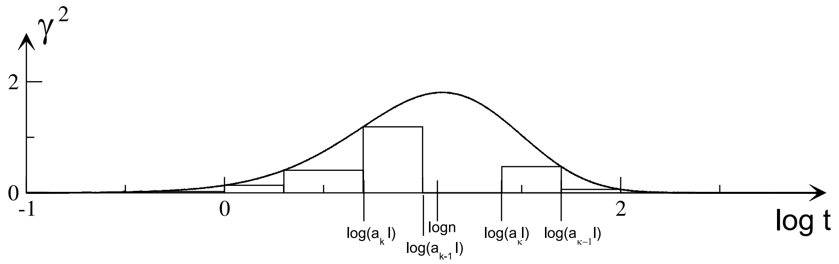
$$\sum_{j=k}^{\infty} \gamma_n^2(a_j l) \cdot \nu_j \leq \sum_{j=k}^{\infty} \gamma_n^2(a_j l) \cdot \nu_{j-1}\frac{\mathfrak{c}_2}{\mathfrak{c}_1},$$

and since $a_{k-1}l \leq n$, the sum on the right–hand–side is less than

$$\int_0^n \gamma_n^2(\alpha)\frac{d\alpha}{\alpha} \cdot \frac{\mathfrak{c}_2}{\mathfrak{c}_1}.$$

Further, let $\kappa$ be the biggest index such that $a_\kappa l \geq n$, supposed it exists (Fig. 5.3.) If $\kappa > 0$, the sum $\sum_{j=0}^{\kappa-1} \gamma_n^2(a_j l)\nu_j$ is majorized by the integral $\int_n^\infty \gamma_n^2(\alpha)\, d\alpha/\alpha$. Further, for indices $j$ between $\kappa$ and $k-1$, the value of $\gamma_n^2(a_j l)$ is not bigger than $\gamma_n^2(n)$, and the sum of $\nu_j$ (equal to the difference between $\log(a_\kappa)$ and $\log(a_k)$) cannot exceed $3\mathfrak{c}_2$. Alltogether, the second inequality in (2.2) is

Figure 5.3: Upper frame–bound




satisfied with

$$B = \frac{\mathfrak{c}_2}{\mathfrak{c}_1} \cdot \int_0^n \gamma_n^2(\alpha) \frac{d\alpha}{\alpha} + \int_n^\infty \gamma_n^2(\alpha) \frac{d\alpha}{\alpha} + 3\mathfrak{c}_2 \cdot \gamma_n^2(n).$$

□

*Remark 1.* For logarithmic equally distanced scales ($\mathfrak{c}_1 = \mathfrak{c}_2$) the maximal distance between $\log(a_\kappa)$ and $\log(a_k)$ (notation as in the last part of the proof) equals $2\mathfrak{c}_1$. Therefore one obtains for the upper bound

$$B = \int_0^\infty \gamma_n^2(\alpha) \frac{d\alpha}{\alpha} + 2\mathfrak{c}_1 \cdot \gamma_n^2(n).$$

*Remark 2.* The sequence of scales as described in the Corollary satisfies also

$$\sum_{a_j \in \mathcal{A}} \frac{(a_j/\tilde{\mathfrak{c}}_1)^p}{[1 + \tilde{\mathfrak{c}}_2(a_j/\tilde{\mathfrak{c}}_1)]^q} \cdot \nu(a_j) < \mathfrak{c} < \infty$$

independently of $\tilde{\mathfrak{c}}_1$ for any positive constant $\tilde{\mathfrak{c}}_2$ and $0 < p < q$. This can be shown in a similar way as in the Corollary through a comparison of the sum with the integral

$$\int_0^\infty \frac{\alpha^p}{(1 + \tilde{\mathfrak{c}}_2 \alpha)^q} \frac{d\alpha}{\alpha}.$$

## 5.2 A condition for wavelet frames

Our aim in this article is to prove that a set $\{g_{y,b},\ (y,b) \in \Lambda\}$, where $\Lambda$ is some countable grid in $\mathbb{H}$, builds a frame. In this section we shall give a sufficient condition for this, using estimations for integrals of reproducing kernels. The idea to consider the behaviour of convolutions with reproducing kernels comes from [12, Section 5 and 6] by Feichtinger and Gröchenig. This technique is used in the proof of the atomic decomposition of functions in general coorbit spaces (more on this topic can be found in [13] and [20]), however the group structure of the parameter space is involved, a property that does not hold in the theory of the wavelet transform over the sphere. Our proof is similar to one given by Holschneider in [22] for wavelet frames over $\mathbb{R} \times \mathbb{R}_+$. First we need a lemma, which is somehow analogous to the Young inequality for $\mathbb{R}^n$.

**Lemma 9** *Denote by $\mathbb{K}$ the space $\mathbb{R}_+ \times \mathbb{R}_+$ with the measure $(\theta\, d\theta, da/a)$. Let $F$ be such a function $\mathbb{H} \times \mathbb{H} \to \mathbb{R}$ that*

$$F(x,a;y,b) = \frac{1}{b^2} \cdot f\left(\frac{\angle(x,y)}{b}, \frac{a}{b}\right), \quad f \in \mathcal{L}^1(\mathbb{K}),$$

*and $T \in \mathcal{L}^p(\mathbb{H})$, $p \geq 1$. Then the following holds*

$$\|F \circ T\|_{\mathcal{L}^p(\mathbb{H})} \leq 2\pi \,\|f\|_{\mathcal{L}^1(\mathbb{K})} \cdot \|T\|_{\mathcal{L}^p(\mathbb{H})},$$

*where the operation $\circ$ is defined by*

$$F \circ T(x,a) = \int_{\mathbb{H}} F(x,a;y,b)\, T(y,b)\; d\omega(y) \frac{db}{b}.$$

*Proof.* Let $R$ be a non-negative function in $\mathcal{L}^q(\mathbb{H})$ with $p^{-1} + q^{-1} = 1$. We may also suppose, that $F$ and $T$ are non-negative. Then

$$\begin{aligned} \langle F \circ T, R\rangle &= \int_{\mathbb{H}} F \circ T(x,a)\, R(x,a)\; d\omega(x) \frac{da}{a} \\ &= \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{1}{b^2} f\left(\frac{\angle(x,y)}{b}, \frac{a}{b}\right) T(y,b)\, R(x,a)\; d\omega(y) \frac{db}{b}\; d\omega(x) \frac{da}{a}. \end{aligned}$$

By change of variables $a/b \mapsto a$ and exchanging the integrals (since all functions are positive, the integrals may only converge absolutely) we obtain

$$\langle F \circ T, R\rangle = \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{1}{b^2} f\left(\frac{\angle(x,y)}{b}, a\right) R(x,ab)\; d\omega(x) \frac{da}{a}\, T(y,b)\; d\omega(y) \frac{db}{b}.$$

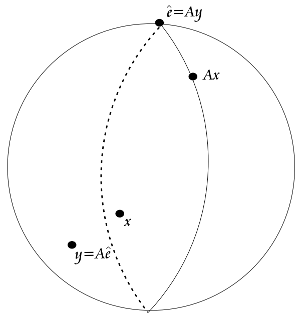
Consider the inner integral with respect to $d\omega(x)$, which we write for simplicity as $\int g(x \cdot y)\, r(x)\, d\omega(x)$. Let $A = A_y$ be the isometry of the sphere which maps $y$ to the North Pole $\hat{e}$ and $\hat{e}$ to $y$ (Fig. 5.4.) Then

$$\begin{aligned} \int g(x \cdot y)\, r(x)\, d\omega(x) &= \int g(Ax \cdot y)\, r(Ax)\, d\omega(Ax) \\ &= \int g(x \cdot A^* y)\, r(Ax)\, d\omega(x) = \int g(x \cdot \hat{e})\, r(Ax)\, d\omega(x) \end{aligned}$$

Now, $Ax$ describes the position of the point $x$ relative to the point $y$ (depending also on the position of the North Pole.) Let $x$ be fixed; by $R_x$ we denote the function $(y,a) \mapsto R(A_y x, a)\, (= r(Ax).)$ Since $A$ was an isometry, we have

$$\int_{\Omega} R_x(y,a)\, d\omega(y) = \int_{\Omega} R(y,a)\, d\omega(y).$$

Figure 5.4: Mapping $A_y$




Then we have (once again exchanging the integrals)

$$\langle F\circ T, R\rangle = \int_{\mathbb{H}}\int_{\mathbb{R}_+}\int_\Omega R_x(y,ab)\,T(y,b)\,d\omega(y)\,\frac{1}{b^2}f\left(\frac{\theta}{b},a\right)\frac{db}{b}\,d\omega(x)\frac{da}{a},$$

where $\theta=\angle(x,\hat{e})$, and further, by the Hölder inequality,

$$\langle F\circ T, R\rangle \le \int_{\mathbb{H}}\int_{\mathbb{R}_+}\|R(\cdot,ab)\|_{\mathcal{L}^q(\Omega)}\|T(\cdot,b)\|_{\mathcal{L}^p(\Omega)}\frac{1}{b^2}f\left(\frac{\theta}{b},a\right)\frac{db}{b}\,d\omega(x)\frac{da}{a}.$$

Now, the integral over $\Omega$ may be estimated as follows:

$$\begin{aligned}\int_\Omega \frac{1}{b^2}f(\theta/b,a)\,d\omega(x) &= 2\pi\int_0^\pi \frac{1}{b^2}f(\theta/b,a)\sin\theta\,d\theta = 2\pi\int_0^{\pi/b} f(\theta,a)\frac{\sin(b\theta)}{b}\,d\theta\\ &\le 2\pi\int_0^{\pi/b} f(\theta,a)\,\theta\,d\theta \le 2\pi\int_0^\infty f(\theta,a)\,\theta\,d\theta = 2\pi\|f(\cdot,a)\|_{\mathcal{L}^1(\mathbb{R}_+,\theta d\theta)},\end{aligned}$$

and therefore, by the Hölder inequality with respect to $db/b$,

$$\begin{aligned}\langle F\circ T, R\rangle &\le 2\pi\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\|f(\cdot,a)\|_{\mathcal{L}^1(\mathbb{R}_+,\theta d\theta)}\|T(\cdot,b)\|_{\mathcal{L}^p(\Omega)}\|R(\cdot,ab)\|_{\mathcal{L}^q(\Omega)}\frac{da}{a}\frac{db}{b}\\ &\le 2\pi\int_{\mathbb{R}_+}\|f(\cdot,a)\|_{\mathcal{L}^1(\mathbb{R}_+,\theta d\theta)}\frac{da}{a}\cdot\|T\|_{\mathcal{L}^p(\mathbb{H})}\|R\|_{\mathcal{L}^q(\mathbb{H})}\\ &= 2\pi\,\|f\|_{\mathcal{L}^1(\mathbb{K})}\|T\|_{\mathcal{L}^p(\mathbb{H})}\|R\|_{\mathcal{L}^q(\mathbb{H})}.\end{aligned}$$

Therefore, we have by the Riesz representation theorem

$$\|F\circ T\|_{\mathcal{L}^p(\mathbb{H})} \le 2\pi\,\|f\|_{\mathcal{L}^1(\mathbb{K})}\|T\|_{\mathcal{L}^p(\mathbb{H})}.$$

Since by assumption all the norms are finite, the exchanges of integrals were justified. □

The theorem below is the main result of this section; a modification of it, Corollary 4 will be used to prove the existence of Poisson wavelet frames.

**Theorem 5** *Let $\{g_a\} \subset \mathcal{L}^2(\mathbb{H})$ be an admissible wavelet family such that it is an analysis–reconstruction pair with itself and let the reproducing kernel*

$$\Pi(x,a;y,b) = \langle g_{x,a}, g_{y,b} \rangle$$

*be such that*

$$\sum_{(y,b)\in\Lambda} |P(x,a;y,b)\,P(y,b;z,c)\,\mu(y,b)| \le \frac{1}{c^2}\tilde{f}\left(\frac{\angle(x,z)}{c}, \frac{a}{c}\right)$$

*for some grid $\Lambda$, weight $\mu$ and $\tilde{f} \in \mathcal{L}^1(\mathbb{K})$. Set*

$$T(x,a;z,c) = \sum_{(y,b)\in\Lambda} \Pi(x,a;y,b)\,\Pi(y,b;z,c)\,\mu(y,b)$$

*(this function is well-defined because of the previous inequality.) Then $\{g_{y,b} : (y,b) \in \Lambda\}$ is a frame with respect to the weight $\mu$ if there exists a constant $\epsilon < 1$ such that*

$$|T(x,a;z,c) - \Pi(x,a;z,c)| < \epsilon\,\frac{1}{c^2} f\left(\frac{\angle(x,z)}{c}, \frac{a}{c}\right), \tag{5.5}$$

*where $f$ is an $\mathcal{L}^1(\mathbb{K})$-function with $\|f\|_1 = 1/(2\pi)$.*

*Proof.* By (2.2) it is enough to show that

$$D := \left| \sum_{(y,b)\in\Lambda} \overline{\mathcal{W}_g r(y,b)}\,\mathcal{W}_g s(y,b)\,\mu(y,b) - \langle r, s\rangle \right| \le \epsilon\,\|r\|_2 \|s\|_2 \tag{5.6}$$

for $r$, $s \in \mathcal{L}^2(\Omega)$. The reproducing kernel equation for the wavelet coefficients yields

$$\sum_{(y,b)\in\Lambda} \overline{\mathcal{W}_g r(y,b)}\,\mathcal{W}_g s(y,b)\,\mu(y,b) = \sum_{(y,b)\in\Lambda} \int_{\mathbb{H}} \overline{\Pi(y,b;x,a)\,\mathcal{W}_g r(x,a)}\; d\omega(x)\frac{da}{a}$$
$$\cdot \int_{\mathbb{H}} \Pi(y,b;z,c)\,\mathcal{W}_g s(z,c)\,d\omega(z)\frac{dc}{c}.$$

Now, we exchnge the summation with the integrals, which we may do because everything converges absolutely, and since $\overline{\Pi(y,b;x,a)} = \Pi(x,a;y,b)$ we obtain that the last expression equals

$$\int_{\mathbb{H}} \int_{\mathbb{H}} T(x,a;z,c)\,\overline{\mathcal{W}_g r(x,a)}\,\mathcal{W}_g s(z,c)\; d\omega(x)\frac{da}{a}\,d\omega(z)\frac{dc}{c}.$$

On the other hand we have for the scalar product of $r$ and $s$

$$\begin{aligned}\langle r, s\rangle &= \int_{\mathbb{H}} \overline{\mathcal{W}_g r(x,a)}\, \mathcal{W}_g s(x,a)\; d\omega(x)\frac{da}{a}\\ &= \int_{\mathbb{H}}\int_{\mathbb{H}} \overline{\mathcal{W}_g r(x,a)}\, \Pi(x,a;z,c)\, \mathcal{W}_g s(z,c)\; d\omega(x)\frac{da}{a}\, d\omega(z)\frac{dc}{c},\end{aligned}$$

and consequently, the difference on the left–hand–side of (5.6) is bounded from above by

$$\int_{\mathbb{H}}\int_{\mathbb{H}} |[T(x,a;z,c) - \Pi(x,a;z,c)]\cdot \mathcal{W}_g s(z,c)|\; d\omega(z)\frac{dc}{c}\, |\mathcal{W}_g r(x,a)|\; d\omega(x)\frac{da}{a}.$$

By the Cauchy–Schwartz inequality for the outer integral we obtain

$$D \le \|(T-P)\circ \mathcal{W}_g s\|_{\mathcal{L}^2(\mathbb{H})}\cdot \|\mathcal{W}_g r\|_{\mathcal{L}^2(\mathbb{H})}$$

and further, by the previous lemma and condition (5.5), we have

$$D \le 2\pi\epsilon\cdot \|f\|_{\mathcal{L}^1(\mathbb{K})}\cdot \|\mathcal{W}_g r\|_{\mathcal{L}^2(\mathbb{H})}\cdot \|\mathcal{W}_g s\|_{\mathcal{L}^2(\mathbb{H})}.$$

Since the wavelet transform is an isometry, the condition (5.6) is satisfied. □

In a similar way, we can perform a discretization of the integral if a semicontinuous wavelet frame is given. More precisely, the following Corollary holds.

**Corollary 4** *Let $\{g_a\} \subset \mathcal{L}^2(\mathbb{H})$ be an admissible wavelet family such that it is an analysis–reconstruction pair with itself. Futher, suppose $\{g_b,\, b\in\mathcal{B}\}$ is a semicontinuous frame with weight $\nu$ and bounds $A \le 1 \le B$ with $B - A < 2$. Let $\Lambda_b \subset \Omega$ be a discrete set of positions for a given scale $b \in \mathcal{B}$ and $\Lambda := \bigcup_{b\in\mathcal{B}} \Lambda_b$. Set*

$$\begin{aligned} S(x,a;z,c) &= \sum_{b\in\mathcal{B}} \int_{\Omega} \Pi(x,a;y,b)\,\Pi(y,b;z,c)\,d\omega(y)\,\nu(b) \quad \textit{and}\\ T(x,a;z,c) &= \sum_{(y,b)\in\Lambda} \Pi(x,a;y,b)\,\Pi(y,b;z,c)\,\mu(y,b)\end{aligned}$$

*for some weight $\mu$, provided both functions are well–defined. If there exists a positive constant $\epsilon < 1 - \sqrt{\frac{B-A}{2}}$ such that*

$$|S(x,a;z,c) - T(x,a;z,c)| < \epsilon\,\frac{1}{c^2} f\left(\frac{\angle(x,z)}{c}, \frac{a}{c}\right), \tag{5.7}$$

*for some $f \in \mathcal{L}^1(\mathbb{K})$ with $\|f\|_1 = 1/(2\pi)$, then $\{g_{y,b},\, (y,b)\in\Lambda\}$ is a weighted frame with the weight $\mu$.*

*Proof.* Similarly as in the previous case we want to show that

$$D := \left| \sum_{(y,b)\in\Lambda} \overline{\mathcal{W}_g r(y,b)}\, \mathcal{W}_g s(y,b)\, \mu(y,b) - \langle r, s\rangle \right| \leq \delta\, \|r\|_2 \|s\|_2$$

with $\delta < 1$. Now, $D$ may be estimated from above by

$$\left| \sum_{(y,b)\in\Lambda} \overline{\mathcal{W}_g r(y,b)}\, \mathcal{W}_g s(y,b)\, \mu(y,b) - \sum_{b\in\mathcal{B}} \int_\Omega \overline{\mathcal{W}_g r(y,b)}\, \mathcal{W}_g s(y,b)\, d\omega(y)\, \nu(b) \right|$$
$$+ \left| \sum_{b\in\mathcal{B}} \int_\Omega \overline{\mathcal{W}_g r(y,b)}\, \mathcal{W}_g s(y,b)\, d\omega(y)\, \nu(b) - \langle r, s\rangle \right|,$$

and for the second summand we obtain the upper bound

$$\sqrt{\frac{B-A}{2}}\, \|r\|_2 \|s\|_2,$$

according to (2.2). The first one is less than or equal to

$$\int_{\mathbb{H}} \int_{\mathbb{H}} |[T(x,a;z,c) - S(x,a;z,c)] \cdot \mathcal{W}_g s(z,c)|\; d\omega(z) \frac{dc}{c}\, |\mathcal{W}_g r(x,a)|\; d\omega(x) \frac{da}{a}.$$

similarly as in the previous case. Therefore, if (5.7) holds, the condition $D \leq \delta\, \|r\|_2 \|s\|_2$ is satisfied (by the Cauchy–Schwartz inequality and Lemma 9), and $\{g_{y,b},\ (y,b) \in \Lambda\}$ is a frame with the weight $\mu$. □

## 5.3 Frames of Poisson wavelets of order $n \geq 2$

Using the results from the Section 5.1 and the last Corollary 4, we are able to prove that Poisson wavelets (of order $n \geq 2$) sampled on a grid of points with spatial density proportional to the squared inverse of the scale $a$ build a weighted frame. However, the construction of such a grid is not so easy as in the case of $\mathbb{R}$ or $\mathbb{R}^2$, where you just take scales $b = 2^j \cdot X$, $j \in \mathbb{Z}$, and positions $y = k \cdot Y b$, $k \in \mathbb{Z}$, respectively $k \in \mathbb{Z} \times \mathbb{Z}$, for some grid constants $X$ and $Y$.

For the wavelet frames on the unit sphere, the scales can be again chosen to be in the geometric progression, see [24] by Holschneider *et al.*, $b = \beta^j \times B$, however this set should be bounded from above, $j \in \mathbb{N}_0$, since $\Omega$ is compact. Remember that the wavelet $g_{x,a}$ was a sum of multipoles localized at the point $e^{-a}x$ inside the sphere. Thus, the set of wavelets on a fixed scale $a$ corresponds to the set of multipoles localized on the sphere of radius $e^{-a}$. The next step is the discretization of positions. In order to obtain a grid whose density does not vary too much over $\Omega$, Holschneider *et al.* adapt the projection of hierarchical subdivision of a cube onto the sphere. More exactly, on each scale $j$, one takes

Figure 5.5: Subdivision of cube facets

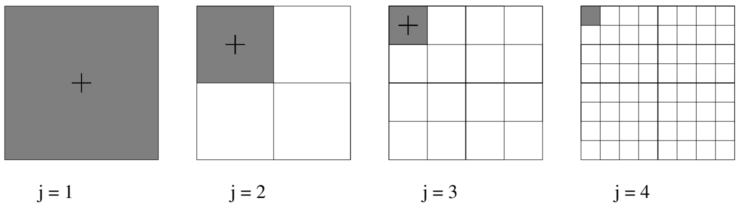



Figure 5.6: Central projection of the point on the cube onto the sphere

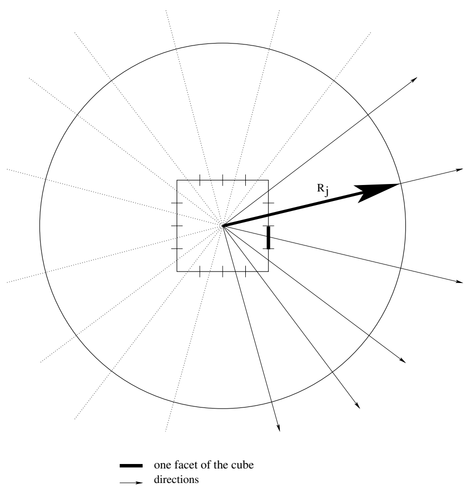
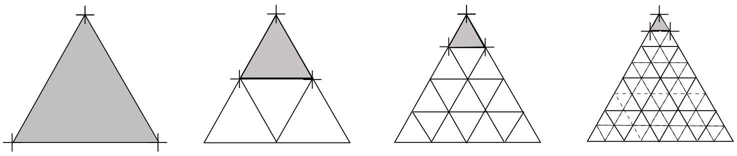


Figure 5.7: Subdivision of icohexaeder facets




a cube centered with respect to the sphere. Each of the sides is divided into $4^j$ squares, see Fig 5.5; the centres of the facets are projected onto the sphere, and they define the $6 \cdot 4^j$ postions for each scale (Fig. 5.6.)
A variation of this concept was used by Chambodut *at al.*, compare [5]. In this case, one takes an icohexaeder instead of the cube. On each scale, its facets are divided into 4 triangles connecting the middle points of the sides. In this case, the $10 \cdot^{j-1} +2$ (4 if $j = 0$) vertices (Fig. 5.7) of the triangles are projected onto the sphere, in the same manner as in the case of the cube. Note that this ensures that the set of points for the level $j$ contains all the points from the level $j-1$.

In our consideration, we generalize this two examples and construct the grid as follows: the set of scales $\mathcal{B} = \{b_j : j \in \mathbb{N}_0\}$ is such that the ratio $b_j/b_{j+1}$ is uniformly bounded form below and from above with the lower bound bigger than 1 (i.e., the sequence $\{b_j\}_{j\in\mathbb{N}_0}$ is decreasing.) On each scale $b$, let $\mathcal{P}_b = \{\mathcal{O}_k^{(b)}, k = 1, 2, \dots, K_b\}$ be a family of simply connected sets (subsets of the sphere) such that
a) their interiors are pairwise disjoint,
b) the sum of all the sets $\mathcal{O}_k^{(b)}$, $k = 1, 2, \dots, K_b$, is equal to $\Omega$,
c) the diameter of each set (measured in central angle) is not bigger than $Yb$ with some proportionality constant $Y$.
Now, choose one point $y = y_k^{(b)}$ from each set $\mathcal{O}_k^{(b)}$ from $\mathcal{P}_b$ and let $\omega(\mathcal{O}_k^{(b)}) \cdot \nu(b)$, where $\omega(\mathcal{O}_k^{(b)})$ is the two-dimensional Lebesgue measure of the set $\mathcal{O}_k^{(b)}$ and $\nu(b)$ the weight of the corresponding semicontinuous frame, be the corresponding weight, which we shall denote by $\mu(y, b)$. We denote the collection of $y$ with the weights $\mu(y, b)$ by $\Lambda_b$. The sum of $\Lambda_b$ over all scales $b$ builds the grid $\Lambda$.

**Theorem 6** *Let $\{g_a\}$, $a \in \mathbb{R}_+$, be a wavelet family such that $\{g_b : b \in \mathcal{B}\}$ is a semicontinuous frame with weight $\nu$ and frame constants $A \leq 1 \leq B$ with $B - A < 2$, and such that*

$$\sum_{b\in\mathcal{B}} \frac{(b/\mathfrak{c}_1)^p}{[1 + \mathfrak{c}_2\,(b/\mathfrak{c}_1)]^q}\,\nu(b) < \mathfrak{c} < \infty \tag{5.8}$$

*uniformly in $\mathfrak{c}_1$, for any positive constant $\mathfrak{c}_2$ and $0 < p < q$. Further, let the* *kernel* $\Pi$ *satisfy*

$$\left.\begin{array}{c}|\Pi(x,a;y,b)|\\ |(a+b)\,\nabla_*\Pi(x,a;y,b)|\end{array}\right\}\leq(ab)^{2+\epsilon}\cdot\begin{cases}\frac{\mathfrak{c}}{(a+b)^{6+2\epsilon}}, & \angle(x,y)\leq\lambda[a+(2-\tilde{\epsilon})b],\\ \frac{\mathfrak{c}}{\angle(x,y)^{6+2\epsilon}}, & \angle(x,y)>\lambda(a+\tilde{\epsilon}b),\end{cases} \tag{5.9}$$

*for* $a,b\leq b_0$ *and for some positive constants* $\mathfrak{c}$, $\lambda$, $\epsilon$ *and* $\tilde{\epsilon}<1/2$, *where* $\nabla_*$ *is the surface gradient with respect to any of the variables* $x$ *or* $y$. *Then there exists a constant* $Y$, *such that for any grid* $\Lambda=\bigcup_{b\in\mathcal{B}}\Lambda_b$ *with* $\Lambda_b$ *constructed according to the above description, the family* $\{g_{y,b}:\,(y,b)\in\Lambda\}$ *builds a frame with weights* $\mu(y,b)$.

*Proof.* According to the Corollary 4 in Section 5.2, we have to show that

$$D=\left|\sum_{(y,b)\in\Lambda}\Pi(x,a;y,b)\Pi(y,b;z,c)\mu(y,b)\right.$$
$$\left.-\sum_{b\in\mathcal{B}}\int_\Omega\Pi(x,a;y,b)\Pi(y,b;z,c)\,d\omega(y)\nu(b)\right|$$

is less than

$$\delta\cdot\frac{1}{c^2}\,f\left(\frac{\angle(x,z)}{c},\frac{a}{c}\right) \tag{5.10}$$

for some $f\in\mathcal{L}^1(\mathbb{K})$ with $\|f\|=\frac{1}{2\pi}$ and $\delta\in\left(0,1-\sqrt{\frac{B-A}{2}}\right)$.
For fixed $(x,a)$, $(z,c)$ and $b$, set $F(y)=\Pi(x,a;y,b)$ and $G(y)=\Pi(z,c;y,b)$. Let $\mathcal{K}_x$ denote the set of points, where $F$ is 'big', i.e., $\mathcal{K}_x=\{y\in\Omega:\,\angle(x,y)\leq\lambda(a+b)\}$. Similarly, denote by $\mathcal{K}_z$ the set '$G$ big', i.e., $\mathcal{K}_z=\{y\in\Omega:\,\angle(y,z)\leq\lambda(c+b)\}$. If the sets $\mathcal{K}_x$ and $\mathcal{K}_z$ are not disjoint, we split the error that one makes by exchanging integration over $\Omega$ by summation over $\{y\in\Omega:\,(y,b)\in\Lambda\}$ into two parts (Fig. 5.8):

– $I_1(b)$: $F(y)$ 'big' or $G(y)$ 'big', i.e., over the set $\mathcal{D}=\mathcal{K}_x\cup\mathcal{K}_z$;

– $I_4(b)$: $F(y)$ 'small' and $G(y)$ 'small', i.e., for $\mathcal{G}=\Omega\backslash(\mathcal{K}_x\cup\mathcal{K}_z)$.

Figure 5.8: Partition of the sphere, $\mathcal{K}_x$ and $\mathcal{K}_z$ not disjoint

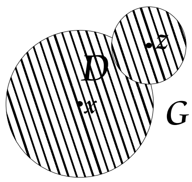
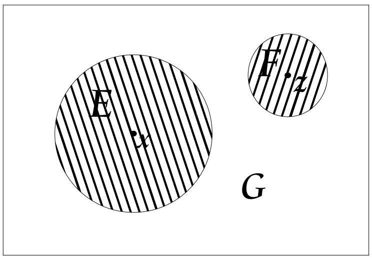


Figure 5.9: Partition of the sphere, $\mathcal{K}_x$ and $\mathcal{K}_z$ disjoint




In the other case, if the sets $\mathcal{K}_x$ and $\mathcal{K}_z$ have an empty section, we consider three parts:

- $I_2(b)$: $F(y)$ 'big', $G(y)$ 'small', i.e., for $\mathcal{E} = \mathcal{K}_x$;
- $I_3(b)$: $F(y)$ 'small', $G(y)$ 'big', i.e., for $\mathcal{F} = \mathcal{K}_z$;
- $I_4(b)$: $F(y)$ 'small', $G(y)$ 'small', i.e., for $\mathcal{G} = \Omega \backslash (\mathcal{K}_x \cup \mathcal{K}_z)$.

Each of the errors may be estimated in the following way: for every set $\mathcal{O} = \mathcal{O}_k^{(b)}$ the difference between the highest and the lowest value of $F(\eta) \cdot G(\eta)$, $\eta \in \mathcal{O}$, is less than or equal to

$$\sup_{\eta \in \mathcal{O}} |\nabla_* [F(\eta) \cdot G(\eta)]| \cdot \operatorname{diam}(\mathcal{O}) \cdot \sup_{\eta \in \mathcal{O}} |G(\eta)|,$$

and hence the difference between $\int_{\mathcal{O}} F(\eta)\, G(\eta)\, d\omega(\eta)\, \nu(b)$ and $F(y)\, G(y)\, \mu(y, b)$ for $y = y_k^{(b)}$ is less than or equal to

$$\left( \sup_{\eta \in \mathcal{O}} |\nabla_* F(\eta)| \cdot \sup_{\eta \in \mathcal{O}} |G(\eta)| + \sup_{\eta \in \mathcal{O}} |F(\eta)| \cdot \sup_{\eta \in \mathcal{O}} |\nabla_* G(\eta)| \right) \cdot \operatorname{diam}(\mathcal{O}) \cdot \mu(y, b). \tag{5.11}$$

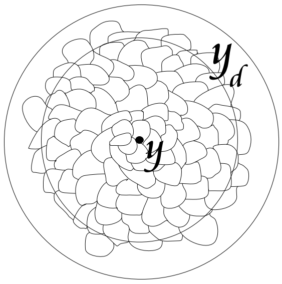
When summing up over all the sets $\mathcal{O}$ that have non–empty section with one of the sets ($\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ or $\mathcal{G}$), we may calculate suprema over the whole set and choose the biggest possible $\operatorname{diam}(\mathcal{O})$ $(= Yb.)$ The sum of $\mu(y, b)$ is then not bigger than the area of $\mathcal{Y}_d$ multiplied by $\nu(b)$, where $\mathcal{Y}_d$ is the $d$–parallel extension of $\mathcal{Y}$, i.e.,

$$\mathcal{Y}_d = \{ \eta \in \Omega |\ \exists y \in \mathcal{Y} : \angle(\eta, y) \leq d \},$$

where $d$ represents the maximal diameter of a set of the partition $\mathcal{P}_b$ (Fig 5.10), and $\mathcal{Y}$ means one of the sets $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ or $\mathcal{G}$.

We introduce the notation $\alpha = a/c$, $\beta = b/c$, $\theta = \angle(x, z)$, $\vartheta = \theta/c$ and $f_j(\alpha, \vartheta) = \sum_b c^2 I_j(b)$ for $j = 1, 2, 3, 4$ and $b \in \mathcal{B}$, but possibly not all the scales.

Figure 5.10: Parallel extension of a set $\mathcal{Y}$




**Part 1)** For $I_1$ we have

$$c^2 I_1(b) \le c^2 \cdot \mathfrak{c} \cdot \left( \frac{1}{a+b} + \frac{1}{c+b} \right) \cdot \frac{(ab)^{2+\epsilon}}{(a+b)^{6+2\epsilon}} \cdot \frac{(bc)^{2+\epsilon}}{(c+b)^{6+2\epsilon}} \cdot Y b \cdot \omega(\mathcal{D}_d) \cdot \nu(b). \quad (5.12)$$

The set $\mathcal{D}$ enclosed in the sum of the spherical circles $\mathcal{K}_x$ and $\mathcal{K}_z$ and hence, the area of $\mathcal{D}_d$ is bounded by twice the area of the bigger spherical circle with radius enlarged by $d$. This is given by $2\pi\lambda^2 \cdot \left(c + (1 + Y/\lambda)\, b\right)^2 \le \mathfrak{c}(c+b)^2$ if $a < c$, respectively $2\pi\lambda^2 \cdot \left(a + (1 + Y/\lambda)\, b\right)^2 \le \mathfrak{c}(a+b)^2$ if $a \ge c$. In the case $\alpha \le 1$, we obtain from (5.12):

$$c^2 I_1(b) \le \mathfrak{c} Y \cdot \frac{\alpha^{2+\epsilon}}{(\alpha+\beta)^{2+\epsilon/2}} \cdot \frac{\beta^{5+3\epsilon/2}}{(\alpha+\beta)^{5+3\epsilon/2}} \cdot \frac{\beta^{\epsilon/2}}{(1+\beta)^{4+2\epsilon}} \cdot \nu(b). \qquad (5.13)$$

The second fraction is smaller than 1, and the last one ensures the summability over $b$, thus, for $\vartheta \le \lambda(\alpha+1)$ we have the estimation

$$f_1 \le \mathfrak{c} Y \cdot \alpha^{\epsilon/2}. \qquad (\textbf{A1})$$

For big $\vartheta$, $\vartheta > \lambda(\alpha+1)$, we use the fact that the sets $\mathcal{K}_x$ and $\mathcal{K}_z$ have a non–empty section only for $b$ such that $\lambda(\alpha+2\beta+1) \ge \vartheta$, i.e., $2(1+\beta) \ge \vartheta/\lambda + 1 - \alpha$, and therefore we may enlarge the last fraction in the estimation (5.13), and write

$$\frac{\beta^{\epsilon/2}}{(1+\beta)^{4+2\epsilon}} \le \frac{\beta^{\epsilon/2}}{(1+\beta)^{2\epsilon}} \cdot \frac{\mathfrak{c}}{[\vartheta + \lambda(1-\alpha)]^4}.$$

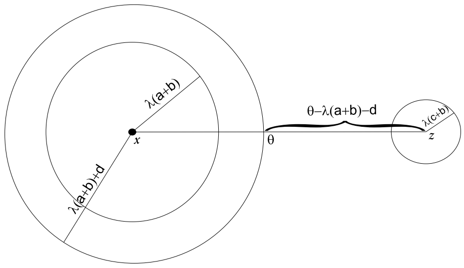
Consequently, we obtain

$$f_1 \le \mathfrak{c} Y \cdot \frac{\alpha^{\epsilon/2}}{[\vartheta + \lambda(1-\alpha)]^4}. \qquad (\textbf{B1})$$

In the other case, $\alpha > 1$, we get

$$c^2 I_1(b) \le \mathfrak{c} Y \cdot \frac{\alpha^{2+\epsilon}}{(\alpha+\beta)^{4+2\epsilon}} \cdot \frac{\beta^{5+2\epsilon}}{(1+\beta)^{7+2\epsilon}} \cdot \nu(b).$$

Figure 5.11: Angular arguments used in Part 2)




For $\vartheta \leq \lambda(1+\alpha)$ we then have

$$f_1 \leq \mathfrak{c}Y \cdot \frac{1}{\alpha^{2+\epsilon}} \tag{\textbf{C1}}$$

and for $\vartheta > \lambda(1+\alpha)$ we write

$$f_1 \leq \mathfrak{c}Y \cdot \frac{1}{(\alpha+\beta)^{2+\epsilon}} \leq \mathfrak{c}Y \cdot \frac{1}{[\vartheta + \lambda(\alpha-1)]^{2+\epsilon}}, \tag{\textbf{D1}}$$

since $2(\alpha+\beta) \geq \vartheta/\lambda + \alpha - 1$.

**Part 2)** In the second case, $I_2(b)$, we consider only the scales for which $\mathcal{K}_x$ and $\mathcal{K}_z$ have an empty section, i.e., $b$ such that $\vartheta > \lambda(\alpha + 2\beta + 1)$. For the error made in the whole set $\mathcal{E}$ we use the formula (5.11) with $\mu(y,b)$ replaced by the area of $\mathcal{E}_d$ (i.e., the area of $(\mathcal{K}_x)_d$) multiplied by $\nu(b)$. Supremum of the modules of $G$ and $\nabla_* G$ is estimated by their values in the point nearest $\mathcal{K}_z$. Since we have to consider all the sets $\mathcal{O}_k^{(b)}$ that have a non–empty section with $\mathcal{E}$, we choose the angular argument in (5.9) to be equal to $\theta - \lambda(a+b) - d$, see Figure 5.11. We have to assume that the maximal diameter of a partition set is less than $\mathfrak{c} \cdot \lambda b$, with some $\mathfrak{c} < 1/2$. For the sake of simplicity, we set $d \leq \lambda b/3$. Alltogether we obtain

$$\begin{aligned} c^2 I_2(b) \leq c^2 \cdot \mathfrak{c} \cdot &\left( \frac{1}{a+b} + \frac{1}{\theta - \lambda(a+4b/3)} \right) \\ &\cdot \frac{(ab)^{2+\epsilon}}{(a+b)^{6+2\epsilon}} \cdot \frac{(bc)^{2+\epsilon}}{[\theta - \lambda(a+4b/3)]^{6+2\epsilon}} \cdot Yb \cdot (a+b)^2 \cdot \nu(b). \end{aligned} \tag{5.14}$$

Further, in the considered range of scales we have $\vartheta/\lambda > \alpha + 2\beta + 1 = \alpha + 4\beta/3 + 1 + 2\beta/3$, and this inequality implies $\vartheta - \lambda(\alpha + 4\beta/3) > [\vartheta + \lambda(2-\alpha)]/3$ as well as $\frac{\mathfrak{c}_1}{c+b} < \frac{1}{\theta - \lambda(a+4b/3)} < \frac{\mathfrak{c}_2}{c+b}$.
For $\alpha \leq 1$, we write the estimation (5.14) in the form

$$c^2 I_2(b) \leq \mathfrak{c}Y \cdot \frac{\alpha^{2+\epsilon}}{(\alpha+\beta)^{1+\epsilon}} \cdot \frac{\beta^{4+\epsilon}}{(\alpha+\beta)^{4+\epsilon}} \cdot \frac{\beta^{1+\epsilon}}{(1+2\beta/3)^{1+2\epsilon}} \cdot \frac{1}{[\vartheta + \lambda(2-\alpha)]^5} \cdot \nu(b),$$

that yields

$$f_2 \le \mathfrak{c}Y \cdot \frac{\alpha}{[\vartheta + \lambda(2-\alpha)]^5}, \tag{\textbf{A2}}$$

and for $\alpha > 1$ we have

$$c^2 I_2(b) \le \mathfrak{c}Y \cdot \frac{\alpha^{2+\epsilon}}{(\alpha+\beta)^{3+\epsilon}} \cdot \frac{\beta^{1+\epsilon}}{(\alpha+\beta)^{1+\epsilon}} \cdot \frac{\beta^{4+\epsilon}}{(1+2\beta/3)^{4+2\epsilon}} \cdot \frac{1}{[\vartheta+\lambda(2-\alpha)]^3} \cdot \nu(b);$$

consequently,

$$f_2 \le \frac{\mathfrak{c}Y}{\alpha\,[\vartheta + \lambda(2-\alpha)]^3}. \tag{\textbf{B2}}$$

**Part 3)** Similarly as in the previous case, we obtain from

$$\begin{aligned} c^2 I_3(b) \le \mathfrak{c}Y & \left( \frac{1}{\theta - \lambda(c+4b/3)} + \frac{1}{c+b} \right) \\ & \cdot \frac{(ab)^{2+\epsilon}}{[\theta-\lambda(c+4b/3)]^{6+2\epsilon}} \cdot \frac{(bc)^{2+\epsilon}}{(c+b)^{6+2\epsilon}} \cdot c^2\, b\,(c+b)^2 \cdot \nu(b) \end{aligned} \tag{5.15}$$

the estimations

$$c^2 I_3(b) \le \mathfrak{c}Y \cdot \alpha^{2+\epsilon} \cdot \frac{1}{[\vartheta+\lambda(2\alpha-1)]^3} \cdot \frac{\beta^{4+2\epsilon}}{(\alpha+2\beta/3)^{4+2\epsilon}} \cdot \frac{\beta}{(1+\beta)^{4+2\epsilon}} \cdot \nu(b),$$

for $a \le c$ and

$$c^2 I_3(b) \le \mathfrak{c}Y \cdot \alpha^{2+\epsilon} \cdot \frac{1}{[\vartheta+\lambda(2\alpha-1)]^6} \cdot \frac{\beta^{2\epsilon}}{(\alpha+2\beta/3)^{2\epsilon}} \cdot \frac{\beta^5}{(1+\beta)^{5+2\epsilon}} \cdot \nu(b)$$

for $a > c$. They yield

$$f_2 \le \mathfrak{c}Y \cdot \frac{\alpha^{2+\epsilon}}{[\vartheta+\lambda(2\alpha-1)]^3} \tag{\textbf{A3}}$$

for $\alpha \le 1$ and

$$f_2 \le \mathfrak{c}Y \cdot \frac{\alpha^{2+\epsilon}}{[\vartheta+\lambda(2\alpha-1)]^6} \tag{\textbf{B3}}$$

for $\alpha > 1$.

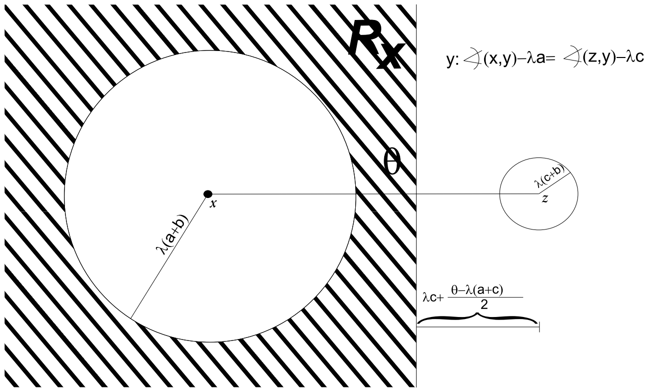
**Part 4)** a) Consider first big $\theta$ and small scales $b$, that is, satisfying the condition $\theta > \lambda(a+2b+c)$. For the points $y$ on the sphere that lie nearer the spherical circle $\mathcal{K}_x$, i.e., elements of the set

$$\mathcal{R}_x := \{y \in \Omega\backslash\mathcal{K}_x : \angle(x,y) - \lambda a \le \angle(z,y) - \lambda c\}, \tag{5.16}$$

compare Figure 5.12, and for one set $\mathcal{O} = \mathcal{O}_k^{(b)}$, we estimate the error using formula (5.11); the terms $\sup_{\eta\in\mathcal{O}} |G(\eta)|$ and $\sup_{\eta\in\mathcal{O}} |\nabla_* G(\eta)|$ may be replaced by the biggest possible value in the $d$–parallel extension of $\mathcal{R}_x$, i.e.

$$\sup_{\eta\in\mathcal{O}} |G(\eta)| \le \frac{\mathfrak{c}\cdot(cb)^{2+\epsilon}}{\theta_z^{6+2\epsilon}} \quad \text{resp.} \quad \sup_{\nabla_*\eta\in\mathcal{O}} |G(\eta)| \le \frac{\mathfrak{c}\cdot(cb)^{2+\epsilon}}{(c+b)\theta_z^{6+2\epsilon}} \tag{5.17}$$

Figure 5.12: Angular arguments used in Part 4a)




with

$$\theta_z = \lambda c + \frac{\theta - \lambda(a+c)}{2} - d \geq \frac{\theta + \lambda(2c-a)}{3} \geq \lambda\left(c + \frac{2}{3}\,b\right).$$

Further, $\sup_{\eta\in\mathcal{O}} |\nabla_{\!*} F(\eta)| \cdot \mu(y,b)$ resp. $\sup_{\eta\in\mathcal{O}} |F(\eta)| \cdot \mu(y,b)$ may be estimated by

$$\frac{(ab)^{2+\epsilon}}{a+b}\int_{\mathcal{O}} \frac{d\omega(y)}{(\angle(x,y)-d)^{6+2\epsilon}} \quad \text{resp.} \quad (ab)^{2+\epsilon}\int_{\mathcal{O}} \frac{d\omega(y)}{(\angle(x,y)-d)^{6+2\epsilon}}$$

multiplied by $\nu(b)$. The bound we obtain for the error is bigger if we sum up over *all* the partition sets having a non empty section with the complement of $\mathcal{K}_x$ (with $\sup_\eta |G(\eta)|$ given by (5.17), a property that does not hold in the whole $(\Omega\backslash\mathcal{K}_x)_d$.) Since $d \leq \lambda b/3$, we obtain

$$\begin{aligned} c^2 I_4^{(x)}(b) \leq c^2 \cdot \mathfrak{c} \cdot &\left(\frac{1}{a+b} + \frac{1}{c+b}\right) \\ &\cdot \int_{\Omega_x} \frac{(ab)^{2+\epsilon}}{(\angle(x,y) - \lambda\, b/3)^{6+2\epsilon}} \cdot \frac{(bc)^{2+\epsilon}}{\theta_z^{6+2\epsilon}} \cdot Y b\, d\omega(y) \cdot \nu(b) \end{aligned}$$

where $I_4^{(x)}(b)$ means the error made in the set $\mathcal{R}_x$ and $\Omega_x$ is the set $\{y \in \Omega : \angle(x,y) \geq \lambda(a + 2b/3)\}$. Denote $\angle(x,y)$ by $\sigma$, then the integral is given by

$$\int_{\lambda(a+2b/3)}^{\pi} \frac{Y a^{2+\epsilon}\, b^{5+2\epsilon}\, c^{2+\epsilon}}{(\sigma - \lambda\, b/3)^{6+2\epsilon}\, \theta_z^{6+2\epsilon}} \sin\sigma\, d\sigma, \tag{5.18}$$

and upon replacing $\sin\sigma$ by $\sigma = (\sigma - \lambda\, b/3) + \lambda\, b/3$ and the upper integration bound $\pi$ by $\infty$, we obtain

$$\begin{aligned} c^2 I_4^{(x)}(b) &\le \mathfrak{c}Y \cdot \left( \frac{1}{a+b} + \frac{1}{c+b} \right) \\ &\quad \cdot \left( \frac{a^{2+\epsilon}\, b^{5+2\epsilon}\, c^{4+\epsilon}}{(a+b/3)^{4+2\epsilon}\, \theta_z^{6+2\epsilon}} + \frac{b}{3} \cdot \frac{a^{2+\epsilon}\, b^{5+2\epsilon}\, c^{4+\epsilon}}{(a+b/3)^{5+2\epsilon}\, \theta_z^{6+2\epsilon}} \right) \cdot \nu(b) \\ &\le \mathfrak{c}Y \cdot \left( \frac{1}{a+b} + \frac{1}{c+b} \right) \cdot \frac{a^{2+\epsilon}\, b^{5+2\epsilon}\, c^{4+\epsilon}}{(a+b/3)^{4+2\epsilon}\, \theta_z^{6+2\epsilon}} \cdot \nu(b) \end{aligned} \tag{5.19}$$

For $\alpha \le 1$ we can write:

$$c^2 I_4^{(x)}(b) \le \mathfrak{c}Y \cdot \frac{\alpha^2}{\alpha+\beta} \cdot \frac{\alpha^\epsilon\, \beta^{4+\epsilon}}{(\alpha+\beta/3)^{4+2\epsilon}} \cdot \frac{\beta^{4+\epsilon}}{\vartheta_z^{1+2\epsilon}} \cdot \frac{1}{\vartheta_z^5} \cdot \nu(b). \tag{5.20}$$

In the second case, $\alpha > 1$, the inequality (5.19) yields

$$c^2 I_4^{(x)}(b) \le \mathfrak{c}Y \cdot \frac{\alpha^{2+\epsilon}}{(\alpha+\beta/3)^{3+\epsilon}} \cdot \frac{\beta^{2+\epsilon}}{(1+\beta)\,(\alpha+\beta/3)^{1+\epsilon}} \cdot \frac{\beta^{3+\epsilon}}{\vartheta_z^{3+2\epsilon}} \cdot \frac{1}{\vartheta_z^3} \cdot \nu(b). \tag{5.21}$$

Analogously for points nearer the other spherical circle, i.e., elements of

$$\mathcal{R}_z := \{ y \in \Omega \backslash \mathcal{K}_z : \angle(x,y) - \lambda a > \angle(z,y) - \lambda c \}, \tag{5.22}$$

we obtain

$$\begin{aligned} c^2 I_4^{(z)}(b) &\le \mathfrak{c} \cdot \left( \frac{1}{a+b} + \frac{1}{c+b} \right) \\ &\quad \cdot \int_{\Omega_z} \frac{(ab)^{2+\epsilon}}{\theta_x^{6+2\epsilon}} \cdot \frac{(bc)^{2+\epsilon}}{\angle(z,y)^{6+2\epsilon}} \cdot c^2 \cdot Y b \, d\omega(y) \cdot \nu(b), \end{aligned} \tag{5.23}$$

where $\Omega_z = \{ y \in \Omega : \angle(z,y) \ge \lambda(c + 2b/3) \}$ and

$$\theta_x = \lambda a + \frac{\theta - \lambda(a+c)}{2} - d \ge \frac{\theta + \lambda(2a-c)}{3} \ge \lambda \left( a + \frac{2}{3}\, b \right)$$

(and $I_4^{(z)}$ is the error made in the set $\mathcal{R}_z$.) The right–hand–side of the inequality (5.23) may be enlarged so that we get

$$c^2 I_4^{(z)}(b) \le \mathfrak{c}Y \cdot \left( \frac{1}{\alpha+\beta} + \frac{1}{1+\beta} \right) \cdot \frac{\alpha^{2+\epsilon} \beta^{5+2\epsilon}}{\vartheta_x^{6+2\epsilon}\, (1+\beta/3)^{4+2\epsilon}} \cdot \nu(b), \tag{5.24}$$

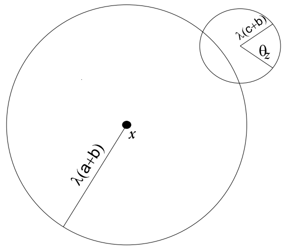
and we write it for $\alpha \le 1$ as

$$c^2 I_4^{(z)}(b) \le \mathfrak{c}Y \cdot \frac{\alpha^{2+\epsilon}}{\vartheta_x^3} \cdot \frac{\beta^{4+2\epsilon}}{(\alpha+\beta)\, \vartheta_x^{3+2\epsilon}} \cdot \frac{\beta}{(1+\beta/3)^{4+2\epsilon}} \cdot \nu(b). \tag{5.25}$$

If $\alpha > 1$, we use the factorization

$$c^2 I_4^{(z)}(b) \le \mathfrak{c}Y \cdot \frac{\alpha^{2+\epsilon}}{\vartheta_x^6} \cdot \frac{\beta^{1+2\epsilon}}{(1+\beta)\, \vartheta_x^{2\epsilon}} \cdot \frac{\beta^4}{(1+\beta/3)^{4+2\epsilon}} \cdot \nu(b). \tag{5.26}$$

Figure 5.13: Angular arguments used in Part 4c)




b) If $\theta > \lambda(a+c)$ and $b$ is such that $\theta \leq \lambda(a+2b+c)$, we estimate the error in a similar way, but we set

$$\theta_x = \lambda(a+b) - d \quad \text{and} \quad \theta_z = \lambda(c+b) - d. \tag{5.27}$$

We obtain again the estimations (5.20), (5.21), (5.25) and (5.26). In the first two of them, the denominator of the third fraction is always bigger than or equal to powered $\lambda(1+2\beta/3)$, and hence it ensures the summability over $b$; the second fraction is not bigger than a constant. In the inequalities (5.25) and (5.26), one can replace the second fraction by a constant, since $\vartheta_x \geq \lambda(\alpha+2\beta/3)$. Further, the estimations

$$\theta_x \geq \frac{\theta + \lambda(2a-c)}{3} \quad \text{and} \quad \theta_z \geq \frac{\theta + \lambda(2c-a)}{3}$$

are also valid for $\theta_x$ and $\theta_z$ defined by (5.27) if the range of scales is bounded by $\theta/\lambda \leq a+2b+c$, which is the case here. Consequently, we obtain from (5.20) and (5.25)

$$f_4 \leq \mathfrak{c}Y \cdot \frac{\alpha}{[\vartheta + \lambda(2-\alpha)]^5} + \mathfrak{c} \cdot \frac{\alpha^{2+\epsilon}}{[\vartheta + \lambda(2\alpha-1)]^3} \tag{\textsf{\textbf{A4}}}$$

for $\alpha \leq 1$ and from (5.21) and (5.26)

$$f_4 \leq \mathfrak{c}Y \cdot \frac{1}{\alpha\,[\vartheta + \lambda(2-\alpha)]^3} + \mathfrak{c} \cdot \frac{\alpha^{2+\epsilon}}{[\vartheta + \lambda(2\alpha-1)]^6} \tag{\textsf{\textbf{B4}}}$$

for $\alpha > 1$.

c) Now, for $\theta \leq \lambda(a+c)$, the spherical circles $\mathcal{K}_x$ and $\mathcal{K}_z$ have a non–empty section for all scales $b$ (Fig. 5.13.) Since $\Omega\backslash(\mathcal{K}_x \cup \mathcal{K}_z) \subseteq \Omega\backslash\mathcal{K}_z$ and $\sup_{\eta \in (\Omega\backslash\mathcal{K}_z)_d} |G(\eta)| = |G(\lambda(c+b)-d)|$, the inequality (5.19) with $\theta_z \geq \lambda(c+2b/3)$:

$$c^2 I_4(b) \leq \mathfrak{c}Y \cdot \left(\frac{1}{\alpha+\beta} + \frac{1}{1+\beta}\right) \cdot \frac{\alpha^{2+\epsilon}\beta^{5+2\epsilon}}{(\alpha+\beta/3)^{4+2\epsilon}(1+2\beta/3)^{6+2\epsilon}} \cdot \nu(b)$$

yields an estimation of the error made in the whole set $I_4$. For $\alpha \leq 1$ we write it as

$$c^2 I_4(b) \leq \mathfrak{c}Y \cdot \frac{\alpha^{2+\epsilon}}{\alpha+\beta} \cdot \frac{\beta^{4+2\epsilon}}{(\alpha+\beta/3)^{4+2\epsilon}} \cdot \frac{\beta}{(1+2\beta/3)^{6+2\epsilon}} \cdot \nu(b)$$

and obtain for the sum over all scales:

$$f_4 \leq \mathfrak{c}Y \cdot \alpha^{1+\epsilon}. \qquad (\mathbf{C4})$$

In the opposite case, $\alpha > 1$, one has

$$c^2 I_4(b) \leq \mathfrak{c}Y \cdot \frac{\alpha^{2+\epsilon}}{(\alpha+\beta/3)^{4+2\epsilon}} \cdot \frac{\beta^{5+2\epsilon}}{(1+\beta)\,(1+2\beta/3)^{6+2\epsilon}} \cdot \nu(b),$$

and consequently

$$f_4 \leq \mathfrak{c}Y \cdot \frac{1}{\alpha^{2+\epsilon}}. \qquad (\mathbf{D4})$$

The following table sorts the obtained estimations:

| | $\vartheta \leq \lambda(\alpha+1)$ | $\vartheta > \lambda(\alpha+1)$ |
|---|---|---|
| $\alpha \leq 1$ | (**A1**) (**C4**) | (**B1**) (**A2**) (**A3**) (**A4**) |
| $\alpha > 1$ | (**C1**) (**D4**) | (**D1**) (**B2**) (**B3**) (**B4**) |

Explicitly, we have

$$\begin{aligned}
f(\alpha,\vartheta) \leq& Y\left(\mathfrak{c}\cdot\alpha^{\epsilon/2} + \mathfrak{c}\cdot\alpha^{1+\epsilon}\right) \qquad \text{for } \alpha \leq 1 \text{ and } \vartheta \leq \lambda(\alpha+1),\\
f(\alpha,\vartheta) \leq& Y\left(\frac{\mathfrak{c}\cdot\alpha^{\epsilon/2}}{[\vartheta+\lambda(1-\alpha)]^4} + \frac{\mathfrak{c}\cdot\alpha}{[\vartheta+\lambda(2-\alpha)]^5} + \frac{\mathfrak{c}\cdot\alpha^{2+\epsilon}}{[\vartheta+\lambda(2\alpha-1)]^3}\right)\\
&\text{for } \alpha \leq 1 \text{ and } \vartheta > \lambda(\alpha+1),\\
f(\alpha,\vartheta) \leq& \frac{\mathfrak{c}Y}{\alpha^{2+\epsilon}} \qquad \text{for } \alpha > 1 \text{ and } \vartheta \leq \lambda(\alpha+1),\\
f(\alpha,\vartheta) \leq& Y\left(\frac{\mathfrak{c}}{[\vartheta+\lambda(\alpha-1)]^{2+\epsilon}} + \frac{\mathfrak{c}}{\alpha\,[\vartheta+\lambda(2-\alpha)]^3} + \frac{\mathfrak{c}\cdot\alpha^{2+\epsilon}}{[\vartheta+(2\alpha-1)]^6}\right)\\
&\text{for } \alpha > 1 \text{ and } \vartheta > \lambda(\alpha+1),
\end{aligned}$$

and hence, $f$ is an $\mathcal{L}^1$–integrable function over $\mathbb{K}$. Since the value of the integral depends linearly on the constant $Y$, it can be arbitrarily small. □

This theorem may be applied to the Poisson wavelets of order $n \geq 2$.

*Remark.* The choice $d \leq \lambda b/3$ does not influence the generality of the statements. If one takes as claimed $d \leq \lambda\left(\frac{1}{2}-\tilde{\delta}\right) b$ for some $\tilde{\delta} > 0$, the resulting integrals change, but they are still convergent.

**Corollary 5** *Let $\{g_a^n\}$, $a \in \mathbb{R}_+$, be a Poisson wavelet family of order $n \geq 2$, normalized such that it is its own reconstruction family, i.e., $\int_{\mathbb{R}_+} |\gamma_n(t)|^2 \frac{dt}{t} = 1$, and $\Lambda$ a grid of points as decribed in the beginning of the section and such that the frame bounds of the corresponding semicontinuous frame satisfy $B - A < 2$. Then $\{g_{y,b}^n : (y,b) \in \Lambda\}$ is a frame with weight $\mu$ for $\mathcal{L}^2(\Omega)$.*

*Remark.* A proper choice of scales ensures that the condition $B - A < 2$ is satisfied, compare formulae in the proof of Corollary 3.
*Proof.* One has to check if the assumptions of the Theorem 6 are satisfied. The set of scales $\mathcal{B}$ is constructed in the same way as in Corollary 3 in Section 5.1, therefore, $\{g_b : b \in \mathcal{B}\}$ is a semicontinuous frame with weight $\nu$ for $\nu(b_j) = \log(b_j/b_{j+1})$; further, by Remark 2 on page 54, the inequality (5.8) is also satisfied.
It remains to check if the estimations on the kernel and its gradient hold. As shown in Section 4.6, the kernel is given by

$$\Pi(x, a; y, b) = \frac{(ab)^n}{(a+b)^{2n}} \, g_{a+b}^{2n} \left(\angle(x, y)\right),$$

and from Section 4.5, Theorem 2 we have

$$\left| g_{a+b}^{2n} \left(\angle(x, y)\right) \right| \leq \mathfrak{c} \cdot \frac{(a+b)^{2n}}{\angle(x, y)^{2n+2}}$$

uniformly in $\angle(x, y)$ for $a + b \in (0, 2 \cdot b_0]$, further

$$\left| g_{a+b}^{2n}(\theta) \right| \leq \frac{\mathfrak{c}}{(a+b)^2}$$

uniformly for $a + b \leq 2b_0$, see Section 4.5, Corollary 1. Since

$$\frac{(ab)^{n-1-\epsilon}}{\theta^{2(n-1-\epsilon)}} \leq \mathfrak{c}$$

for $\theta \geq \lambda(a + \epsilon b)$ and $\epsilon < 1$ and

$$\frac{(ab)^{n-1-\epsilon}}{(a+b)^{2(n-1-\epsilon)}} \leq 1$$

for $\epsilon < 1$, the inequalities (5.9) are satisfied for the kernel.
For the surface gradient of the kernel we have

$$\nabla_* \Pi(x, a; y, b) = \frac{(ab)^n}{(a+b)^{2n}} \, \nabla_* \, g_{a+b}^{2n} \left(\angle(x, y)\right),$$

and since the longitudinal derivative of the wavelet vanishes, the absolute value of the gradient $\nabla_* \, g_{a+b}^{2n} \left(\angle(x, y)\right)$ for any $x$, $y$ is less than or equal to the absolute value of the derivative with respect to $\theta$ for $\theta = \angle(x, y)$. The Theorem 3, Section 4.5 yields

$$|\nabla_* \Pi(x, a; y, b)| \leq \mathfrak{c} \cdot \frac{(ab)^n}{(a+b)^{2n}} \, \frac{(a+b)^{2n}}{\theta^{2n+3}},$$

uniformly in $\theta$ for $a+b \le 2b_0$ and consequently

$$|(a+b)\nabla_* \Pi(x,a;y,b)| \le \mathfrak{c} \cdot \frac{(ab)^n}{\theta^{2n+2}} \le \mathfrak{c} \cdot \frac{(ab)^{2+\epsilon}}{\theta^{6+2\epsilon}}$$

for $\theta \ge \lambda(a+\epsilon b)$. On the other hand, we have

$$\left|\nabla_* g_{a+b}^{2n}\left(\angle(x,y)\right)\right| \le \frac{\mathfrak{c}}{(a+b)^3},$$

compare Corollary 2 in Section 4.5, and therefore

$$|(a+b)\nabla_*\Pi(x,a;y,b)| \le \mathfrak{c} \cdot \frac{(ab)^n}{(a+b)^{2n+2}} \le \mathfrak{c} \cdot \frac{(ab)^{2+\epsilon}}{(a+b)^{6+2\epsilon}}.$$

Thus, the inequalities (5.9) are satisfied and Theorem 6 applies to Poisson wavelets of order $n \ge 3$. □

## 5.4 Frames of $1^{\text{st}}$ and $2^{\text{nd}}$–order Poisson wavelets

We are also able to construct countable sets of points $(y,b) \in \Omega \times \mathbb{R}_+$ such that $\{g_{y,b}^1\}$, respectively $\{g_{y,b}^2\}$, is a weighted frame. In both cases, the points lie tighter, the maximal diameter $d$ of a single partition set scales with $b$ like $b^{2+\delta}$ (for first–order wavelets) or $b^{1+\delta}$ (for second–order wavelets) for some positive $\delta$. More precisely, let again the set of scales $\mathcal{B} = \{b_j : j \in \mathbb{N}_0\}$ be such that the ratio $b_j/b_{j+1}$ is uniformly bounded form below and from above with the lower bound bigger than 1; on each scale $b$, the family let $\mathcal{P}_b = \{\mathcal{O}_k^{(b)},\ k = 1,2,\dots,K_b\}$ be a family of simply connected subsets of the sphere satisfies
a) their interiors of $\mathcal{O}_k^{(b)}$ are pairwise disjoint,
b) the sum of all the sets $\mathcal{O}_k^{(b)}$, $k = 1,2,\dots,K_b$, is equal to $\Omega$,
c) the diameter of each set is not bigger than
— $Yb^{2+\delta}$ in the case $n = 1$ or
— $Yb^{1+\delta}$ in the case $n = 2$
with some proportinality constant $Y$ and some positive $\delta$.
We choose one point $y = y_k^{(b)}$ from each set $\mathcal{O}_k^{(b)}$ and set $\omega(\mathcal{O}_k^{(b)}) \cdot \nu(b)$ to be the corresponding weight, denoted by $\mu(y,b)$. The collection of $y$ with the weights $\mu(y,b)$ is denoted by $\Lambda_b$. The sum of $\Lambda_b$ over all scales $b$ builds the grid $\Lambda$.

**Theorem 7** *Let $\{g_a\}$, $a \in \mathbb{R}_+$, be a wavelet family such that $\{g_b : b \in \mathcal{B}\}$ is a semicontinuous frame with weight $\nu$ and frame constants $A \le 1 \le B$ with $B - A < 2$, and satisfying*

$$\sum_{b\in\mathcal{B}} b^{\delta/2}\nu(b) < \mathfrak{c} < \infty. \tag{5.28}$$

*Further, let the kernel* $\Pi$ *satisfy*

$$\left.\begin{array}{c}|\Pi(x,a;y,b)|\\ |(a+b)\,\nabla_*\Pi(x,a;y,b)|\end{array}\right\}\le ab\cdot\begin{cases}\frac{\mathfrak{c}}{(a+b)^4}, & \angle(x,y)\le\lambda[a+(2-\tilde\epsilon)b],\\ \frac{\mathfrak{c}}{\angle(x,y)^4}, & \angle(x,y)>\lambda(a+\tilde\epsilon b),\end{cases} \tag{5.29}$$

*for* $a,b\le b_0$ *and for some positive constants* $\mathfrak{c}$, $\lambda$ *and* $\tilde\epsilon<1/2$. *Then, there exists a constant* $Y$, *such that for any grid* $\Lambda=\bigcup_{b\in\mathcal{B}}\Lambda_b$ *with* $\Lambda_b$ *constructed according to the above description (with* $d\sim Yb^{2+\delta}$*), the set* $\{g_{y,b}:\,(y,b)\in\Lambda\}$ *builds a frame with weights* $\mu(y,b)$.
*For wavelets with the property*

$$\left.\begin{array}{c}|\Pi(x,a;y,b)|\\ |(a+b)\,\nabla_*\Pi(x,a;y,b)|\end{array}\right\}\le ab\cdot\begin{cases}\frac{\mathfrak{c}}{(a+b)^6}, & \angle(x,y)\le\lambda[a+(2-\tilde\epsilon)b],\\ \frac{\mathfrak{c}}{\angle(x,y)^6}, & \angle(x,y)>\lambda(a+\tilde\epsilon b),\end{cases} \tag{5.30}$$

*the required density of points is lower, i.e., the above statement holds for grids with* $d\sim Yb^{1+\delta}$.

*Proof.* Again, we have to prove that

$$D=\Bigg|\sum_{(y,b)\in\Lambda}\Pi(x,a;y,b)\Pi(y,b;z,c)\mu(y,b)$$
$$-\sum_{b\in\mathcal{B}}\int_\Omega\Pi(x,a;y,b)\Pi(y,b;z,c)\,d\omega(y)\nu(b)\Bigg|$$

is less than

$$\tilde\delta\cdot\frac{1}{c^2}\,f\left(\frac{\angle(x,z)}{c},\frac{a}{c}\right) \tag{5.31}$$

for some $f\in\mathcal{L}^1(\mathbb{K})$ with $\|f\|=\frac{1}{2\pi}$ and $\tilde\delta\in\left(0,1-\sqrt{\frac{B-A}{2}}\right)$.
We shall use the same notation as in the last section. Since the set of scales $b$ is bounded from above, there exists a constant $\mathfrak{c}$ such that $d\le\mathfrak{c}\cdot b$. Therefore, we may use the same estimations of areas of the sets $\mathcal{Y}_d$ as in the previous case. The only modification in the estimation of the error for a fixed scale $b$ is in $\operatorname{diam}(\mathcal{O})$ (and obviously in the constant.) When summing over all scales, we use the fact that $c^{\delta/2}\le\mathfrak{c}$, respectively, $a^{\delta/2}\le\mathfrak{c}$.

**Part 1)** For the set $I_1$ and in the case $\alpha\le1$, the modified (i.e., with $\epsilon=-1$ and multiplied by $b^{1+\delta}$) estimation (5.13) can be written as

$$\begin{aligned}c^2\,I_1(b)&\le\mathfrak{c}Y\cdot\frac{ab^{4+\delta}c^3}{(a+b)^5(c+b)^2}\cdot\nu(b)\\ &=\mathfrak{c}Y\cdot c^{1+\delta/2}\cdot\frac{a\,c^{2-\delta/2}}{(a+b)^{1-\delta/2}(c+b)^2}\cdot\frac{b^{4+\delta/2}}{(a+b)^{4+\delta/2}}\cdot b^{\delta/2}\cdot\nu(b).\end{aligned}$$

For $\vartheta \le \lambda(\alpha+1)$ we therefore obtain

$$f_1 \le \mathfrak{c}Y \cdot \alpha^{\delta/2}, \tag{\textbf{A1a}}$$

and in the other case, $\vartheta > \lambda(\alpha+1)$, we have

$$f_1 \le \mathfrak{c}Y \cdot \frac{\alpha}{[\vartheta+\lambda(\alpha-1)]^{1-\delta/2}[\vartheta+\lambda(1-\alpha)]^2}, \tag{\textbf{B1a}}$$

since $2(1+\beta) > \vartheta/\lambda + 1 - \alpha$ and $2(\alpha+\beta) > \vartheta/\lambda + \alpha - 1$, analogously as for $n \ge 3$.
For $\alpha > 1$ we have:

$$\begin{aligned} c^2\, I_1(b) &\le \mathfrak{c}Y \cdot \frac{ab^{4+\delta}c^3}{(a+b)^2(c+b)^5} \cdot \nu(b) \\ &= \mathfrak{c}Y \cdot a^{1+\delta/2} \cdot \frac{c^{2+\delta/2}}{a^{\delta/2}(a+b)^2} \cdot \frac{b^{4+\delta/2}c^{1-\delta/2}}{(c+b)^5} \cdot b^{\delta/2} \cdot \nu(b) \\ &= \mathfrak{c}Y \cdot \frac{ac^{1/2}}{(a+b)^{3/2}} \cdot \frac{b^3}{(a+b)^{1/2}(c+b)^{5/2}} \cdot \frac{c^{5/2}}{(c+b)^{5/2}} \cdot b^{1+\delta} \cdot \nu(b), \end{aligned}$$

that yields

$$f_1 \le \frac{\mathfrak{c}Y}{\alpha^{2+\delta/2}} \tag{\textbf{C1a}}$$

for $\vartheta \le \lambda(\alpha+1)$ and

$$f_1 \le \mathfrak{c}Y \cdot \frac{1}{\alpha^{1/2}\,[\vartheta+\lambda(1-\alpha)]^{5/2}} \tag{\textbf{D1a}}$$

for $\vartheta > \lambda(\alpha+1)$.

For wavelets satisfying (5.30), the inequality for $\alpha \le 1$ reads (we take $\epsilon = 0$ and factor $b^\delta$)

$$\begin{aligned} c^2\, I_1(b) &\le \mathfrak{c}Y \cdot \frac{a^2 b^{5+\delta} c^4}{(a+b)^7(c+b)^4} \cdot \nu(b) \\ &= \mathfrak{c}Y \cdot c^{1+\delta/2} \cdot \frac{a^2c^{4-\delta/2}}{(a+b)^{2-\delta/2}(c+b)^4} \cdot \frac{b^{5+\delta/2}}{(a+b)^{5+\delta/2}} \cdot b^{\delta/2} \cdot \nu(b), \end{aligned}$$

and therefore

$$f_1 \le \mathfrak{c}Y \cdot \alpha^{\delta/2}, \tag{\textbf{A1b}}$$

for $\vartheta \le \lambda(\alpha+1)$ and

$$f_1 \le \mathfrak{c}Y \cdot \frac{\alpha^{\delta/2}}{[\vartheta+\lambda(1-\alpha)]^4}, \tag{\textbf{B1b}}$$

for $\vartheta > \lambda(\alpha+1)$.

Similarly, for $\alpha > 1$ we obtain

$$\begin{aligned} c^2 I_1(b) &\le \mathfrak{c}Y \cdot \frac{a^2 b^{5+\delta} c^4}{(a+b)^4 (c+b)^7} \cdot \nu(b) \\ &= \mathfrak{c}Y \cdot a^{\delta/2} \cdot \frac{a^{2-\delta/2} c^{2+\delta/2}}{(a+b)^4} \cdot \frac{b^{5+\delta/2} c^{2-\delta/2}}{(c+b)^7} \cdot b^{\delta/2} \cdot \nu(b) \\ &= \mathfrak{c}Y \cdot \frac{a^2 c}{(a+b)^3} \cdot \frac{b^5}{(a+b)(c+b)^4} \cdot \frac{c^3}{(c+b)^3} \cdot b^{\delta} \cdot \nu(b), \end{aligned}$$

and consequently

$$f_1 \le \frac{\mathfrak{c}Y}{\alpha^{2+\delta/2}} \qquad (\textbf{C1b})$$

for $\vartheta \le \lambda(\alpha+1)$, and

$$f_1 \le \mathfrak{c}Y \cdot \frac{1}{\alpha\,[\vartheta + \lambda(1-\alpha)]^3} \qquad (\textbf{D1b})$$

for $\vartheta > \lambda(\alpha+1)$.

**Part 2)** For the estimation of $f_2$ we rewrite the inequality (5.14) as

$$\begin{aligned} c^2 I_2(b) &\le \mathfrak{c}Y \cdot \left( \frac{1}{a+b} + \frac{1}{\theta - \lambda(a+4b/3)} \right) \cdot \frac{a b^{4+\delta} c^3}{(a+b)^2 [\theta - \lambda(a+4b/3)]^4} \cdot \nu(b) \\ &\le \begin{cases} \mathfrak{c}Y \cdot \frac{ac^3}{[\theta-\lambda(a+4b/3)]^4} \cdot \frac{b^3}{(a+b)^3} \cdot b^{1+\delta} \cdot \nu(b), & \text{for } \alpha \le 1, \\ \mathfrak{c}Y \cdot \frac{ac^{1/2}}{(a+b)^{3/2}} \cdot \frac{b^3 c^{5/2}}{(a+b)^{1/2}[\theta-\lambda(a+4b/3)]^5} \cdot b^{1+\delta} \cdot \nu(b), & \text{for } \alpha > 1, \end{cases} \end{aligned}$$

and therefore we have

$$f_2 \le \mathfrak{c}Y \cdot \frac{\alpha}{[\vartheta + \lambda(2-\alpha)]^4}. \qquad (\textbf{A2a})$$

for $\alpha \le 1$ and

$$f_2 \le \frac{\mathfrak{c}Y}{\alpha^{1/2} \cdot [\vartheta + \lambda(2-\alpha)]^{5/2}}. \qquad (\textbf{B2a})$$

for $\alpha > 1$.

If the inequality (5.30) is satisfied, we can write

$$\begin{aligned} c^2 I_2(b) &\le \mathfrak{c}Y \cdot \left( \frac{1}{a+b} + \frac{1}{\theta - \lambda(a+4b/3)} \right) \cdot \frac{a^2 b^{5+\delta} c^4}{(a+b)^4 [\theta - \lambda(a+4b/3)]^6} \cdot \nu(b) \\ &\le \begin{cases} \mathfrak{c}Y \cdot \frac{a^2c^4}{[\theta-\lambda(a+4b/3)]^6} \cdot \frac{b^5}{(a+b)^5} \cdot b^{\delta} \cdot \nu(b), & \text{for } \alpha \le 1, \\ \mathfrak{c}Y \cdot \frac{a^2 c^2}{(a+b)^3} \cdot \frac{b^5 c^3}{(a+b)[\theta-\lambda(a+4b/3)]^7} \cdot b^{\delta} \cdot \nu(b), & \text{for } \alpha > 1, \end{cases} \end{aligned}$$

and consequently

$$f_2 \le \mathfrak{c}Y \cdot \frac{\alpha^2}{[\vartheta + \lambda(2-\alpha)]^6}. \qquad (\textbf{A2b})$$

for $\alpha \leq 1$ and

$$f_2 \leq \frac{\mathfrak{c}Y}{\alpha \cdot [\vartheta + \lambda(2-\alpha)]^3}. \tag{\textbf{B2b}}$$

for $\alpha > 1$.

**Part 3)** The next estimation (an equivalent of (5.15)) reads

$$c^2 I_3(b) \leq \mathfrak{c}Y \cdot \left( \frac{1}{\theta - \lambda(c+4b/3)} + \frac{1}{c+b} \right) \cdot \frac{ab^{4+\delta}c^3}{[\theta - \lambda(c+4b/3)]^4\,(c+b)^2} \cdot \nu(b)$$
$$\leq \begin{cases} \mathfrak{c}Y \cdot \frac{ac^{3/2}}{[\theta-\lambda(c+4b/3)]^{5/2}} \cdot \frac{b^3 c^{3/2}}{[\theta-\lambda(c+4b/3)]^{5/2}(c+b)^2} \cdot b^{1+\delta} \cdot \nu(b), & \text{for } \alpha \leq 1, \\ \mathfrak{c}Y \cdot \frac{ac^3}{[\theta-\lambda(c+4b/3)]^4} \cdot \frac{b^3}{(c+b)^3} \cdot b^{1+\delta} \cdot \nu(b), & \text{for } \alpha > 1, \end{cases}$$

and yields

$$f_2 \leq \mathfrak{c}Y \cdot \frac{\alpha}{[\vartheta + \lambda(2\alpha - 1)]^{5/2}} \tag{\textbf{A3a}}$$

for $\alpha \leq 1$ and

$$f_2 \leq \mathfrak{c}Y \cdot \frac{\alpha}{[\vartheta + \lambda(2\alpha - 1)]^4} \tag{\textbf{B3a}}$$

for $\alpha > 1$.

Analogously, for wavelets satisfying (5.30) we have

$$c^2 I_3(b) \leq \mathfrak{c}Y \cdot \left( \frac{1}{\theta - \lambda(c+4b/3)} + \frac{1}{c+b} \right) \cdot \frac{a^2 b^{5+\delta}c^4}{[\theta - \lambda(c+4b/3)]^6\,(c+b)^4} \cdot \nu(b)$$
$$\leq \begin{cases} \mathfrak{c}Y \cdot \frac{a^2 c}{[\theta-\lambda(c+4b/3)]^3} \cdot \frac{b^5 c^3}{[\theta-\lambda(c+4b/3)]^4(c+b)^4} \cdot b^{\delta} \cdot \nu(b), & \text{for } \alpha \leq 1, \\ \mathfrak{c}Y \cdot \frac{a^2c^4}{[\theta-\lambda(c+4b/3)]^6} \cdot \frac{b^5}{(c+b)^5} \cdot b^{\delta} \cdot \nu(b), & \text{for } \alpha > 1, \end{cases}$$

and hence

$$f_2 \leq \mathfrak{c}Y \cdot \frac{\alpha^2}{[\vartheta + \lambda(2\alpha - 1)]^3} \tag{\textbf{A3b}}$$

for $\alpha \leq 1$ and

$$f_2 \leq \mathfrak{c}Y \cdot \frac{\alpha^2}{[\vartheta + \lambda(2\alpha - 1)]^6} \tag{\textbf{B3b}}$$

for $\alpha > 1$.

**Part 4)** The error estimation in the set $\mathcal{D}$ for $\theta$ bigger than $\lambda(a+c)$ is given by (5.19) and (5.24) with $\epsilon = -1$ and the right–hand side multiplied by $b^{1+\delta}$:

$$c^2 I_4^{(x)}(b) \leq \mathfrak{c}Y \cdot \left( \frac{1}{\alpha+\beta} + \frac{1}{1+\beta} \right) \cdot \frac{\alpha\beta^3}{(\alpha+\beta/3)^2\,\vartheta_z^4} \cdot b^{1+\delta} \cdot \nu(b)$$

and

$$c^2 I_4^{(z)}(b) \leq \mathfrak{c}Y \cdot \left( \frac{1}{\alpha+\beta} + \frac{1}{1+\beta} \right) \cdot \frac{\alpha\beta^3}{\vartheta_x^4\,(1+\beta/3)^2} \cdot b^{1+\delta} \cdot \nu(b).$$

Since
$$\vartheta_z \geq \begin{cases} \frac{\vartheta+\lambda(2-\alpha)}{3} \\ \frac{\lambda(3+2\beta)}{3} \end{cases} \quad \text{and} \quad \vartheta_x \geq \begin{cases} \frac{\vartheta+\lambda(2\alpha-1)}{3} \\ \frac{\lambda(3\alpha+2\beta)}{3} \end{cases},$$
we can write for the sum of the above inequalities in the case $\alpha \leq 1$:
$$c^2 I_4(b) \leq \mathfrak{c}Y \cdot \left( \frac{\alpha}{[\vartheta+\lambda(2-\alpha)]^4} \cdot \frac{\beta^3}{(\alpha+\beta)(\alpha+\beta/3)^2} \right.$$
$$\left. + \frac{\alpha}{[\vartheta+\lambda(2\alpha-1)]^{5/2}} \cdot \frac{\beta^3}{[\lambda(\alpha+2/3\,\beta)]^{3/2}(\alpha+\beta)(1+\beta/3)^2} \right) \cdot b^{1+\delta} \cdot \nu(b)$$
and therefore we obtain
$$f_4 \leq \mathfrak{c}Y \cdot \frac{\alpha}{[\vartheta+\lambda(2-\alpha)]^4} + \mathfrak{c}Y \cdot \frac{\alpha}{[\vartheta+\lambda(2\alpha-1)]^{5/2}}. \qquad (\textbf{A4a})$$
On the other hand we have for $\alpha > 1$:
$$c^2 I_4(b) \leq \mathfrak{c}Y \cdot \left( \frac{\alpha}{(\alpha+\beta/3)^{3/2}} \cdot \frac{\beta^3}{(1+\beta)(\alpha+\beta/3)^{1/2}\,[\lambda(1+2/3\,\beta)]^{3/2}} \cdot \frac{1}{[\vartheta+\lambda(2-\alpha)]^{5/2}} \right.$$
$$\left. + \frac{\alpha}{[\vartheta+\lambda(2\alpha-1)]^4} \cdot \frac{\beta^3}{(1+\beta)(1+\beta/3)^2} \right) \cdot b^{1+\delta} \cdot \nu(b),$$
and hence
$$f_4 \leq \mathfrak{c}Y \cdot \frac{1}{\alpha^{1/2}\,[\vartheta+\lambda(2-\alpha)]^{5/2}} + \mathfrak{c}Y \cdot \frac{\alpha}{[\vartheta+\lambda(2\alpha-1)]^4}. \qquad (\textbf{B4a})$$
If $\theta \leq \lambda(a+c)$, we take the modified estimation (5.19) with $\theta_z \geq \lambda(c+2b/3)$ of the error made in the set $\Omega\backslash\mathcal{K}_z \supseteq \Omega\backslash(\mathcal{K}_x \cup \mathcal{K}_z)$:
$$c^2 I_4(b) \leq \mathfrak{c}Y \cdot \left( \frac{1}{a+b} + \frac{1}{c+b} \right) \cdot \frac{ab^3c^4}{(a+b/3)^2(c+2b/3)^4} \cdot b^{1+\delta} \cdot \nu(b)$$
$$= \mathfrak{c}Y \cdot a^{1+\delta/2} \cdot \left( \frac{b^{\delta/2}c^{1-\delta/2}}{a+b} + \frac{b^{\delta/2}c^{1-\delta/2}}{c+b} \right) \cdot \frac{c^{2+\delta/2}}{a^{\delta/2}(a+b/3)^2} \cdot \frac{b^4}{(c+2b/3)^4} \cdot b^{\delta/2} \cdot \nu(b)$$
and get for $\alpha \leq 1$
$$f_4 \leq \mathfrak{c}Y \cdot \frac{\alpha}{(1+2\beta/3)^4} \cdot \frac{\beta^3}{(\alpha+\beta)(\alpha+\beta/3)^2} \leq \mathfrak{c}Y \cdot \alpha. \qquad (\textbf{C4a})$$
In the opposite case, $\alpha > 1$, this inequality yields
$$f_4 \leq \mathfrak{c}Y \cdot \frac{1}{\alpha^{\delta/2}(\alpha+\beta/3)^2} \leq \frac{\mathfrak{c}Y}{\alpha^{2+\delta/2}}. \qquad (\textbf{D4a})$$

For wavelets satisfying (5.30) we have for $\theta > \lambda(a+c)$:
$$c^2 I_4^{(x)}(b) \leq \mathfrak{c}Y \cdot \left( \frac{1}{\alpha+\beta} + \frac{1}{1+\beta} \right) \cdot \left( \frac{\alpha^2\beta^4}{(\alpha+\beta/3)^4\,\vartheta_z^6} + \frac{\alpha^2\beta^4}{\vartheta_x^6\,(1+\beta/3)^4} \right) \cdot b^{1+\delta} \cdot \nu(b)$$

and therefore, for $\alpha \le 1$:

$$c^2 I_4(b) \le \mathfrak{c}Y \cdot \Bigg( \frac{\alpha}{[\vartheta + \lambda(2-\alpha)]^6} \cdot \frac{\alpha\beta^4}{(\alpha+\beta)(\alpha+\beta/3)^4} + \frac{\alpha^2}{[\vartheta+\lambda(2\alpha-1)]^3} \cdot \frac{\beta^4}{(\alpha+2/3\,\beta)^3(\alpha+\beta)(1+\beta/3)^4} \Bigg) \cdot b^{1+\delta} \cdot \nu(b),$$

further

$$f_4 \le \mathfrak{c}Y \cdot \frac{\alpha}{[\vartheta+\lambda(2-\alpha)]^6} + \mathfrak{c}Y \cdot \frac{\alpha^2}{[\vartheta+\lambda(2\alpha-1)]^3}. \qquad (\mathbf{A4b})$$

For $\alpha > 1$ we obtain from

$$c^2 I_4(b) \le \mathfrak{c}Y \cdot \Bigg( \frac{\alpha^2}{(\alpha+\beta/3)^4} \cdot \frac{\beta^4}{(1+\beta)\,[\lambda(1+2/3\,\beta)]^3} \cdot \frac{1}{[\vartheta+\lambda(2-\alpha)]^3} + \frac{\alpha^2}{[\vartheta+\lambda(2\alpha-1)]^6} \cdot \frac{\beta^4}{(1+\beta)(1+\beta/3)^4} \Bigg) \cdot b^{1+\delta} \cdot \nu(b),$$

the estimation

$$f_4 \le \mathfrak{c}Y \cdot \frac{1}{\alpha^2[\vartheta+\lambda(2-\alpha)]^3} + \mathfrak{c}Y \cdot \frac{\alpha^2}{[\vartheta+\lambda(2\alpha-1)]^6}. \qquad (\mathbf{B4b})$$

Further, for $\theta \le \lambda(a+c)$ we obtain

$$\begin{aligned} c^2 I_4(b) &\le \mathfrak{c}Y \cdot \left( \frac{1}{\alpha+\beta} + \frac{1}{1+\beta} \right) \cdot \frac{\alpha^2\beta^4}{(\alpha+\beta/3)^4 \cdot (1+2\beta/3)^6} \cdot b^{1+\delta} \cdot \nu(b) \\ &\le \mathfrak{c}Y \cdot a \cdot \left( \frac{1}{\alpha+\beta} + \frac{1}{1+\beta} \right) \cdot \frac{\alpha\beta^5}{(\alpha+\beta/3)^4 \cdot (1+2\beta/3)^6} \cdot b^{\delta} \cdot \nu(b), \end{aligned}$$

and hence

$$f_4 \le \mathfrak{c}Y \cdot \frac{\alpha^2}{(\alpha+\beta)} \cdot \frac{\beta^4}{(1+2\beta/3)^6(\alpha+\beta/3)^4} \le \mathfrak{c}Y \cdot \alpha \qquad (\mathbf{C4b})$$

for $\alpha \le 1$ and

$$f_4 \le \mathfrak{c}Y \cdot \frac{\alpha}{(\alpha+\beta/3)^4} \cdot \frac{\beta^5}{(1+\beta)(1+2\beta/3)^6} \le \frac{\mathfrak{c}Y}{\alpha^2}. \qquad (\mathbf{D4b})$$

for $\alpha > 1$.

Alltogether, we obtain for wavelets with (5.29):

$$f(\alpha,\vartheta) \leq Y\left(\mathfrak{c}\cdot\alpha^{\delta/2} + \mathfrak{c}\cdot\alpha\right) \qquad \text{for } \alpha \leq 1 \text{ and } \vartheta \leq \lambda(\alpha+1),$$

$$f(\alpha,\vartheta) \leq Y\cdot\left(\frac{\mathfrak{c}\cdot\alpha}{[\vartheta+\lambda(\alpha-1)]^{1-\delta/2}[\vartheta+\lambda(1-\alpha)]^2} + \frac{\mathfrak{c}\cdot\alpha}{[\vartheta+\lambda(2-\alpha)]^4} + \frac{\mathfrak{c}\cdot\alpha}{[\vartheta+\lambda(2\alpha-1)]^{5/2}}\right)$$
$$\text{for } \alpha \leq 1 \text{ and } \vartheta > \lambda(\alpha+1),$$

$$f(\alpha,\vartheta) \leq \frac{\mathfrak{c}Y}{\alpha^{2+\delta/2}} \qquad \text{for } \alpha > 1 \text{ and } \vartheta \leq \lambda(\alpha+1),$$

$$f(\alpha,\vartheta) \leq Y\left(\frac{\mathfrak{c}}{\alpha^{1/2}\,[\vartheta+\lambda(1-\alpha)]^{5/2}} + \frac{\mathfrak{c}}{\alpha^{1/2}\,[\vartheta+\lambda(2-\alpha)]^{5/2}} + \frac{\mathfrak{c}\cdot\alpha}{[\vartheta+(2\alpha-1)]^4}\right)$$
$$\text{for } \alpha > 1 \text{ and } \vartheta > \lambda(\alpha+1),$$

and for wavelets satisfying (5.30):

$$f(\alpha,\vartheta) \leq Y\left(\mathfrak{c}\cdot\alpha^{\delta/2} + \mathfrak{c}\cdot\alpha\right) \qquad \text{for } \alpha \leq 1 \text{ and } \vartheta \leq \lambda(\alpha+1),$$

$$f(\alpha,\vartheta) \leq Y\left(\frac{\mathfrak{c}\cdot\alpha^{\delta/2}}{[\vartheta+\lambda(1-\alpha)]^3} + \frac{\mathfrak{c}\cdot\alpha+\mathfrak{c}\cdot\alpha^2}{[\vartheta+\lambda(2-\alpha)]^6} + \frac{\mathfrak{c}\cdot\alpha^2}{[\vartheta+\lambda(2\alpha-1)]^3}\right)$$
$$\text{for } \alpha \leq 1 \text{ and } \vartheta > \lambda(\alpha+1),$$

$$f(\alpha,\vartheta) \leq \frac{\mathfrak{c}Y}{\alpha^{2+\delta/2}} + \frac{\mathfrak{c}Y}{\alpha^3} \qquad \text{for } \alpha > 1 \text{ and } \vartheta \leq \lambda(\alpha+1),$$

$$f(\alpha,\vartheta) \leq Y\left(\frac{\mathfrak{c}}{\alpha\,[\vartheta+\lambda(1-\alpha)]^3} + \frac{\mathfrak{c}}{\alpha\,[\vartheta+\lambda(2-\alpha)]^3}\right.$$
$$\left. + \frac{\mathfrak{c}}{\alpha^2\,[\vartheta+\lambda(2-\alpha)]^3} + \frac{\mathfrak{c}\cdot\alpha^2}{[\vartheta+(2\alpha-1)]^6}\right)$$
$$\text{for } \alpha > 1 \text{ and } \vartheta > \lambda(\alpha+1).$$

In both cases, $f$ is an $\mathcal{L}^1$–integrable function over $\mathbb{K}$. The value of the integral depends linearly on the constant $Y$, and therefore it can be arbitrarily small.□

This theorem applies to the Poisson wavelets of order $n = 1$ (estimation (5.29)) and $n = 2$ (estimation (5.30)).

# Chapter 6

# Outlook

As already stated in the beginning of this chapter, further inverstigations are needed in order to find some good grids in $\Omega \times \mathbb{R}_+$ and the corresponding frame bounds (compare also [24] and [5].) Moreover, one can try verify the existence of discrete frames in the case of general (non–zonal) spherical wavelets, and investigate their properties.

# Bibliography


[1] J.-P. Antoine, I. Bogdanova, L. Jacques, M. Morvidone, and P. Vandergheynst, *Stereographic wavelet frames on the sphere*, Appl. Comput. Harm. Anal. **19** (2005), no. 2, 223–252.

[2] J.-P. Antoine and P. Vandergheyns, *Wavelets on the n–sphere and related manifolds*, J. Math. Phys. **39** (1998), 3987–4008.

[3] ______, *Wavelets on the 2–sphere: a group–theoretical approach*, Appl. Comput. Harm. Anal. **7** (1999), 262–291.

[4] P. Cerejeiras, M. Ferreira, U. Kähler, and F. Sommen, *Continuous wavelet transform and wavelet frames on the sphere using Clifford analysis*, www.pisharp.org/dspace/handle/2052/118, 2006.

[5] A. Chambodut, I. Panet, M. Mandea, M. Diament, M. Holschneider, and O. Jamet, *Wavelet frames: an alternative of spherical harmonics representation of potential fields*, Gephys. J. Int. **163** (2005), 875–899.

[6] O. Christensen, *An introduction to frames and Riesz bases*, Birkhäuser, Boston Basel Berlin, 2002.

[7] O. Christensen and T.K. Jensen, *An introduction to the theory of bases, frames, and wavelets*, http://citeseer.ist.psu.edu/christensen99introduction.html.

[8] M. Conrad and J. Prestin, *Multiresolution on the sphere*, www.math.muluebeck.de/prestin/ps/mingle.ps, 2002.

[9] S. Dahlke, W. Dahmen, E. Schmitt, and I. Weinreich, *Multiresolution analysis and wavelets on $\mathcal{S}^2$ and $\mathcal{S}^3$*, Numer. Funct. Anal. Optim. **2** (1995), 19–41.

[10] S. Dahlke, G. Teschke, and G. Steidl, *Coorbit spaces and Banach frames on homogenous spaces with applications to analyzing functions on spheres*, Adv. Computat. Math. **21** (2004), no. 1–2, 147–180.

[11] R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), no. 2, 341–366.

[12] H. Feichtinger and K.H. Gröchenig, *Banach spaces related to integrable group representations and their atomic decompositions, I*, J. Funct. Anal. **86** (1989), 307–340.

[13] ———, *Banach spaces related to integrable group representations and their atomic decompositions. Part II*, Mh. Math. **108** (1989), 129–148.

[14] W. Freeden, T. Gervens, and M. Schreiner, *Constructive approximation on the sphere*, Clanderon Press, Oxford, 1998.

[15] W. Freeden, O. Glockner, and R. Litzenberger, *A general Hilbert space approach to wavelets and its application in geopotential determination*, Numer. Funct. Anal. and Optimiz. **20** (1999), 853–879.

[16] W. Freeden and M. Schreiner, *Orthogonal and non–orthogonal multiresolution analysis, scale discrete and exact fully discrete wavelet transform on the sphere*, Constr. Approx **14** (1998), 493–515.

[17] W. Freeden and U. Windheuser, *Combined spherical harmonic and wavelet expansion — A future concept in Earth's gravitational determination*, Appl. Comput. Harm. Anal. **4** (1997), 1–37.

[18] J. Göttelmann, *Locally supported wavelets on manifolds with applications to the 2D sphere*, Appl. Comput. Harm. Anal. **7** (1999), 1–33.

[19] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products*, Academic Press, San Diego, 2000.

[20] P. Gröbner, *Bancharäume glatter Funktionen und Zerlegungsmethoden*, Ph.D. thesis, Universität Wien, 1983.

[21] H. Groemer, *Geometric applications of Fourier series and spherical harmonics*, Cambridge Unversity Press, Camridge, 1996.

[22] M. Holschneider, *Wavelets. An analysis tool*, Clarendon Press, Oxford, 1995.

[23] ———, *Continuous wavelet transform on the sphere*, J. Math. Phys. **37** (1996), 4156–4165.

[24] M. Holschneider, A. Chambodut, and M. Mandea, *From global to regional analysis of the magnetic field on the sphere using wavelet frames*, Phys. Earth Planet. Inter. **4175** (2003), 1–18.

[25] K. Jänich, *Analysis für Physiker und Ingenieure*, Springer–Verlag, Berlin Heidelberg New York, 2001.

[26] W. Magnus and F. Oberhettinger, *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik*, Springer–Verlag, Berlin Göttingen Heidelberg, 1948.

[27] B. Rubin, *Fractional integrals and potentials*, Addison–Wesley/Longman, Essex, 1996.

[28] ———, *Spherical Radon transform and related wavelet transforms*, Appl. Comput. Harm. Anal. **5** (1998), 202–215.

[29] P. Schröder and W. Sweldens, *Spherical wavelets: Efficientls representing functions on the sphere*, Proceedings of the 22nd annual conference on computer graphics and interactive techniques, 1995, pp. 161–172.

[30] ———, *Spherical wavelets: Texture processing*, http://netlib.bell-labs.com/whowim/papers/texture.pdf, 1995.

[31] I. Weinreich, *Biortogonale Wavelets auf der Sphäre*, Shaker, Aachen, 1997.

[32] ———, *A construction of $\mathcal{C}^1$–wavelets on the two–dimensional sphere*, Appl. Comput. Harm. Anal. **10** (2001), 1–26.